Section Summary: 2.6

a. Definitions

- A function $y = f(x)$ may be defined **implicitly**, by an equation $D(x, y) = E(x, y)$. Moving both functions to one side of the equation and defining $D(x, y) - E(x, y) = g(x, y)$, we see that this is equivalent to $g(x, y) = 0$. That is, rather than write $y = f(x)$ **explicitly**, we write the relationship between the two in another way. For example, the function $y = x^2$ can be written explicitly, or as $y - x^2 = 0$. In the second case, it's not clear which is the dependent and which the independent variable.

  It's important to note that there may be multiple functions which are the solution of an implicit equation. For example, $y = \sqrt{x}$ is one solution of the implicit equation $y^2 = x$, or $y^2 - x = 0$ (the other is $y = -\sqrt{x}$).

  Another example is the **folium of Descartes**, which is the graph (solution(s)) of the implicit equation
  $$x^3 + y^3 = 6xy$$
  Here the equation is perfectly symmetric in x and y: it's not at all clear which variable is dependent, and which independent. By the way, this **symmetry in** x **and** y means that the solution curve must be **symmetric about the line** $y = x$. See Figure 4, p. 159.

- **Implicit differentiation**: begin with an equation involving an independent variable x and dependent variable y. Since both sides are equal, their derivatives are equal. Differentiate both sides of the equation by x, using all of the ordinary rules (including the chain rule). Then solve the resulting equation for the derivative y'.

Example: Compute $\frac{dy}{dx}$ implicitly if
$$e^{3x-4y} = x^2 - x$$
Since the two sides are equal, their derivatives must be equal:
$$\frac{d}{dx}e^{3x-4y} = \frac{d}{dx}(x^2 - x)$$
and, using the chain rule and the cool property of exponentials,
$$e^{3x-4y}\frac{d}{dx}(3x - 4y) = 2x - 1$$
Then
$$e^{3x-4y}(3 - 4\frac{dy}{dx}) = 2x - 1$$
so
$$3 - 4\frac{dy}{dx} = \frac{2x - 1}{e^{3x-4y}}$$
and
$$\frac{dy}{dx} = -\frac{1}{4}\left(\frac{2x - 1}{e^{3x-4y}} - 3\right)$$
All done!

b. Summary

Implicit equations are different than the explicit ones we have encountered most often to this point: we haven't "solved" for one of the variables in terms of the other (often we can not solve to find a unique function - there may be more than one, as in the solution of

$$y^2 + x^2 = 1$$

– the equation of a circle of radius 1). The graph of the circle clearly fails the vertical line test, so this equation actually defines two different functions: one for the upper semi-circle, and one for the lower semi-circle.

We can go ahead and differentiate both sides of the equation to get a new equation involving the derivative of the variable treated as the dependent variable. It actually produces two derivative functions: one for the upper semi-cicle, and one for the lower (so we need to choose the proper derivative).