Lab 10

MAT 229, Spring 2021

Exercises to submit

Exercise 1

Consider the sequence given by the formula $\{a_k\}_{k=1}^{\infty} = \left\{\frac{(k!)^2}{(2k)!}\right\}_{k=1}^{\infty}$. (The parentheses are important here. Make sure you have them in the correct places.)

```
In[ ]:= Clear[a]
        a[k_] := (k!)^2/((2 k)!)
```

a. Plot the first 20 terms of this sequence. From this plot, what do you think is the value of $\lim_{k\to\infty} \frac{(k!)^2}{(2k)!}$?

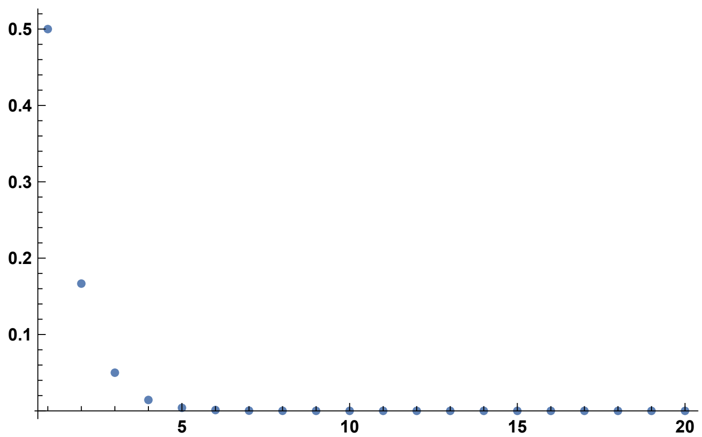
In[∘]:=
```
npts = 20
MatrixForm[Table[{n, N[a[n]]}, {n, 1, npts}]]
ListPlot[Table[{n, a[n]}, {n, 1, npts}], PlotRange → All]
```

Out[∘]= 20

Out[∘]//MatrixForm=

1	0.5
2	0.166667
3	0.05
4	0.0142857
5	0.00396825
6	0.00108225
7	0.000291375
8	0.0000777001
9	0.0000205677
10	5.41254×10^{-6}
11	1.41757×10^{-6}
12	3.69801×10^{-7}
13	9.61483×10^{-8}
14	2.49273×10^{-8}
15	6.44673×10^{-9}
16	1.66367×10^{-9}
17	4.28521×10^{-10}
18	1.10191×10^{-10}
19	2.82923×10^{-11}
20	7.25444×10^{-12}

Out[∘]=



b. Plot the first 20 partial sums $S_n = \sum_{k=1}^{n} \frac{(k!)^2}{(2k)!}$. Based on this plot, do you think the series $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ converges or not? If you think it converges, estimate its value.

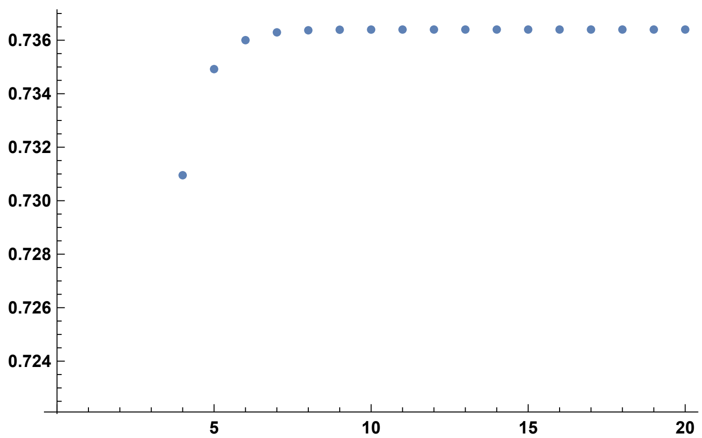
In[∘]:=
```
Clear[s]
s[n_] := Sum[a[k], {k, 1, n}]
npts = 20
MatrixForm[Table[{n, N[s[n]]}, {n, 1, npts}]]
ListPlot[Table[{n, s[n]}, {n, 1, npts}]]
```

Out[∘]= 20

Out[∘]//MatrixForm=

1	0.5
2	0.666667
3	0.716667
4	0.730952
5	0.734921
6	0.736003
7	0.736294
8	0.736372
9	0.736393
10	0.736398
11	0.736399
12	0.7364
13	0.7364
14	0.7364
15	0.7364
16	0.7364
17	0.7364
18	0.7364
19	0.7364
20	0.7364

Out[∘]=



I think it converges, to something around 0.7364....

Exercise 2

Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the alternating harmonic series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$.

In[∘]:=
```
Clear[a]
a[k_] := 1.0 / k
```

a. The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

a.a. Compute the partial sums for this series **as decimal values**:

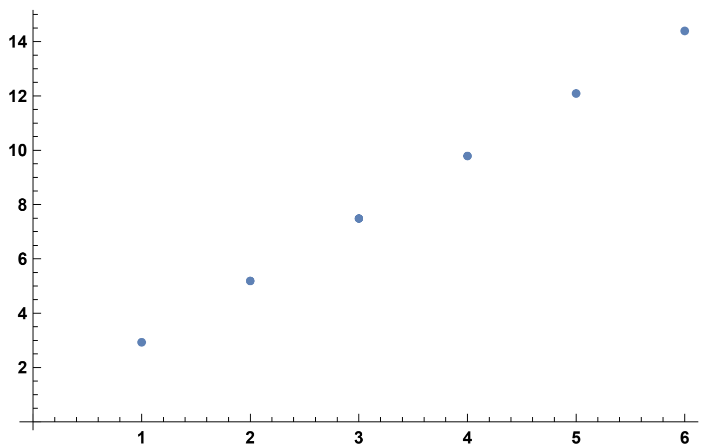
$S_{10}, S_{100}, S_{1000}, S_{10\,000}, S_{100\,000}, S_{1\,000\,000}$.

In[•]:=
```
Clear[s]
s[n_] := Sum[a[k], {k, 1, n}]
npts = 6;
pairs = Table[{n, s[10^n]}, {n, 1, npts}];
MatrixForm[pairs]
ListPlot[pairs]
```

Out[•]//MatrixForm=

1	2.92897
2	5.18738
3	7.48547
4	9.78761
5	12.0901
6	14.3927

Out[•]=



a.b. Estimate by how much the partial sums go up when we increase the number of terms in the partial sum by a factor of 10.

I can eyeball it, and see (rise over run) that it's rising by about 2.3 for every unit increase in power of 10. I can use non-linear regression to be more specific (but I wouldn't expect you to!:):

In[•]:=
```
Clear[b, m]
NonlinearModelFit[pairs, b + m x, {b, m}, x]
```

Out[•]= FittedModel[0.612459 + 2.29512 x]

this model says that the slope is, indeed, around 2.30....:)

b. The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

b.a. Compute the partial sums for this series

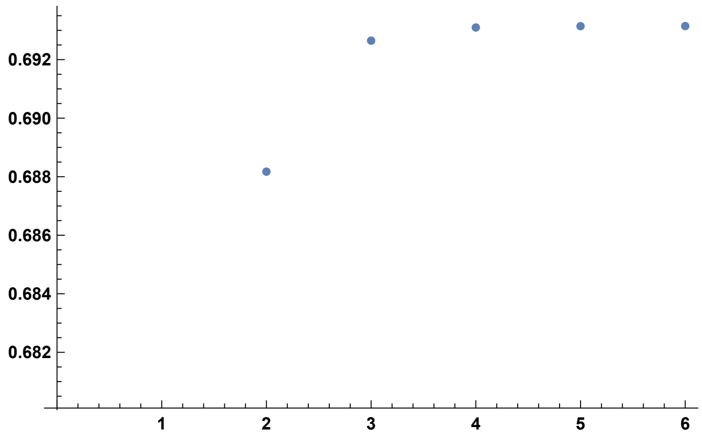
$S_{10}, S_{100}, S_{1000}, S_{10\,000}, S_{100\,000}, S_{1\,000\,000}$.

In[]:=
```
Clear[s]
s[n_] := Sum[(-1) ^ (k + 1) a[k], {k, 1, n}]
npts = 6;
altpairs = Table[{n, s[10^n]}, {n, 1, npts}];
MatrixForm[altpairs]
ListPlot[altpairs]
```

Out[]//MatrixForm=

1	0.645635
2	0.688172
3	0.692647
4	0.693097
5	0.693142
6	0.693147

Out[]=



b.b. To 4 decimal places, what does the alternating harmonic series converge to?

Sure looks like ln(2):

In[]:=
```
N[Log[2], 20]
```

Out[]= 0.69314718055994530942

Exercise 3

Geometric series have the form $\sum_{k=0}^{\infty} a\, r^k$. It converges if and only if $|r| < 1$. If it does converge it converges to $\frac{a}{1-r}$.

a. The geometric series $\sum_{k=0}^{\infty} (0.5)^k$ converges to $\frac{1}{1-0.5} = 2$. Create a plot of the partial sums of this series. Use it to help you zero in on the first partial sum that is within 0.001 of the series value.

In[•]:=
```
Clear[a, s]
a[k_] := (0.5)^k
s[n_] := Sum[a[k], {k, 0, n}]
npts = 12;
MatrixForm[Table[{n, N[s[n]]}, {n, 0, npts}]]
```

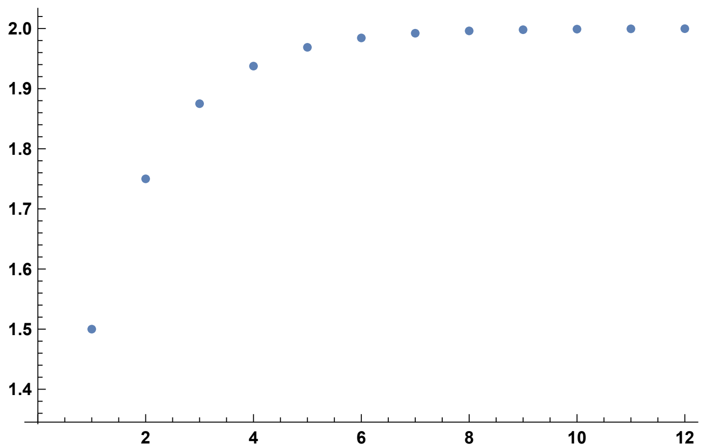
Out[•]//MatrixForm=

0	1.
1	1.5
2	1.75
3	1.875
4	1.9375
5	1.96875
6	1.98438
7	1.99219
8	1.99609
9	1.99805
10	1.99902
11	1.99951
12	1.99976

a.a. Plot the partial sums as points until you see where they level off.

In[•]:=
```
ListPlot[Table[{n, s[n]}, {n, 0, npts}]]
```

Out[•]=



a.b. Use whatever means you have to zoom in around that leveling off to determine where the partial sums start getting within 0.0001 of the convergent value.

It's really S_{15}, since we're adding up the first 15 terms; but it runs from 0 to 14:

In[◦]:=
```
npts = 14;
MatrixForm[Table[{n, N[s[n]]}, {n, 0, npts}]]
```

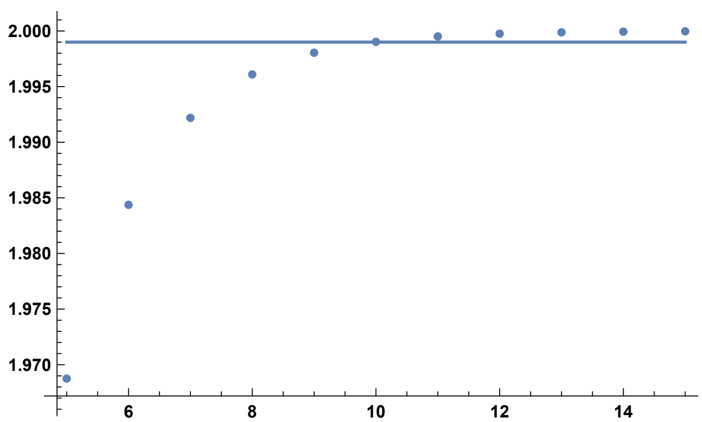
Out[◦]//MatrixForm=

0	1.
1	1.5
2	1.75
3	1.875
4	1.9375
5	1.96875
6	1.98438
7	1.99219
8	1.99609
9	1.99805
10	1.99902
11	1.99951
12	1.99976
13	1.99988
14	1.99994

In[◦]:=
```
npts = 15
Show[
 ListPlot[Table[{n, s[n]}, {n, npts - 10, npts}]],
 Plot[1.999, {n, npts - 10, npts}]
 , PlotRange → All
]
```

Out[◦]= 15

Out[◦]=



b. The geometric series $\sum_{k=0}^{\infty} (0.95)^k$ converges to $\frac{1}{1-0.95} = 20$. Determine the first partial sum that is within 0.0001 of the series value.

In[•]:=
```
Clear[a, s]
a[k_] := (0.95)^k
s[n_] := Sum[a[k], {k, 0, n}]
npts = 248;
MatrixForm[Table[{n, s[n]}, {n, npts - 20, npts}]]
```

Out[•]//MatrixForm=

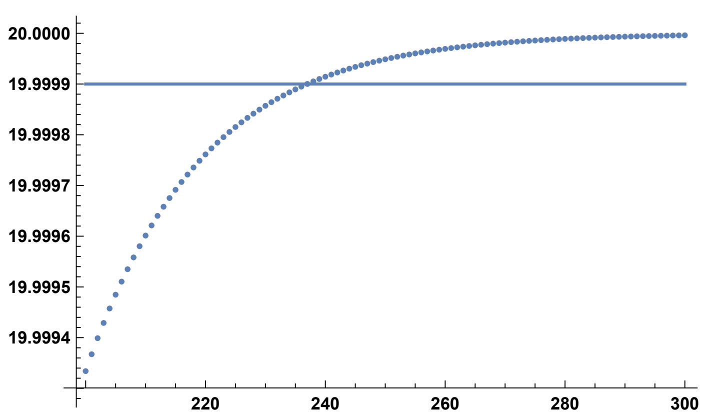
228	19.9998
229	19.9998
230	19.9999
231	19.9999
232	19.9999
233	19.9999
234	19.9999
235	19.9999
236	19.9999
237	19.9999
238	19.9999
239	19.9999
240	19.9999
241	19.9999
242	19.9999
243	19.9999
244	19.9999
245	19.9999
246	19.9999
247	19.9999
248	19.9999

b.a. Plot the partial sums as points until you see where they level off.

In[•]:=
```
npts = 300
Show[
 ListPlot[Table[{n, s[n]}, {n, npts - 100, npts}]],
 Plot[19.9999, {n, npts - 100, npts}]
 , PlotRange → All
]
```

Out[•]= 300

Out[•]=

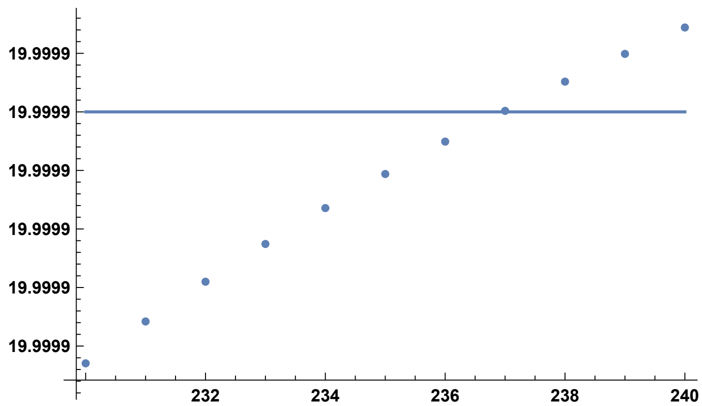


b.b. Use whatever means you have to zoom in around that leveling off to determine where the partial sums start getting within 0.0001 of the convergent value.

In[ ]:=
```
npts = 240
Show[
 ListPlot[Table[{n, s[n]}, {n, npts - 10, npts}]],
 Plot[19.9999, {n, npts - 10, npts}]
 , PlotRange → All
]
```

Out[ ]= 240

Out[ ]=



Looks like it's n=237, but it might be n=238.

Exercise 4

Let the sequence $\{a_k\}_{k=1}^{\infty}$ be given by $\left\{(-1)^{k(k+1)/2}\,\frac{k!}{10^k}\right\}_{k=1}^{\infty}$.

In[ ]:= $a[k\_] := (-1)^{k(k+1)/2}\,\frac{k!}{10^k}$

a. Create a list of the first 10 terms of this sequence.

In[ ]:=
```
npts = 10;
MatrixForm[Table[{n, N[a[n]]}, {n, 1, npts}]]
```

Out[ ]//MatrixForm=

1	-0.1
2	-0.02
3	0.006
4	0.0024
5	-0.0012
6	-0.00072
7	0.000504
8	0.0004032
9	-0.00036288
10	-0.00036288

a.a. What is the pattern with the signs?

They alternate, but in pairs, 2 at a time: -,-,+,+, etc.

a.b. Do you have a guess on what is the value of $\lim_{k\to\infty} a_k$?

I'm guessing that factorial will ultimately kick the exponential's hinder! That's what I'm guess-

ing! So this guy's going to diverge.

b. Create a list of the first 20 terms of this sequence.

In[ ]:=
```
npts = 20;
MatrixForm[Table[{n, N[a[n]]}, {n, 1, npts}]]
```

Out[ ]//MatrixForm=

1	-0.1
2	-0.02
3	0.006
4	0.0024
5	-0.0012
6	-0.00072
7	0.000504
8	0.0004032
9	-0.00036288
10	-0.00036288
11	0.000399168
12	0.000479002
13	-0.000622702
14	-0.000871783
15	0.00130767
16	0.00209228
17	-0.00355687
18	-0.00640237
19	0.0121645
20	0.024329

b.a. Does the pattern with the signs continue?

You betcha!

b.b. Do you still have the same guess about the value of $\lim_{k\to\infty} a_k$?

I think that someone's in for a licking, and it looks like an exponential to me! The factorial is going to beat up on it.

c. Create a list of the first 30 terms of this sequence.

```
In[∘]:= npts = 30;
MatrixForm[Table[{n, N[a[n]]}, {n, 1, npts}]]
```

Out[∘]//MatrixForm=

1	-0.1
2	-0.02
3	0.006
4	0.0024
5	-0.0012
6	-0.00072
7	0.000504
8	0.0004032
9	-0.00036288
10	-0.00036288
11	0.000399168
12	0.000479002
13	-0.000622702
14	-0.000871783
15	0.00130767
16	0.00209228
17	-0.00355687
18	-0.00640237
19	0.0121645
20	0.024329
21	-0.0510909
22	-0.1124
23	0.25852
24	0.620448
25	-1.55112
26	-4.03291
27	10.8889
28	30.4888
29	-88.4176
30	-265.253

c.a. Does the pattern with the signs continue?

Yep!

c.b. Do you still have the same guess about the value of $\lim_{k\to\infty} a_k$?

Oh my, yes. Things are going south for that poor little exponential, dominated by that mean ol' factorial function.

d. What is true about $\sum_{k=1}^{\infty} a_k$? Give reasons for your answer.

It's divergent; but won't tend to either positive or negative infinity because of the sign-switching.