Lab 12

MAT 229, Spring 2021

Today's lab is primarily about the ratio and the root tests for the convergence of a series.

Your job: to explore these examples of uses (or **attempted** uses) of these tests.

Review

Absolute convergence test

If $\sum_{k}^{\infty} \left| b_k \right|$ converges then $\sum_{k}^{\infty} b_k$ converges.

Root and Ratio tests:

Given any series $\sum_{k}^{\infty} b_k$: **evaluate the limit** of either/or

- the ratio of successive terms (**ratio** test), ignoring any signs:

  $L = \lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right|$

- the k^{th} root, ignoring any signs (**root** test), of the k^{th} term:

  $L = \lim_{k\to\infty} \sqrt[k]{\left| b_k \right|}$.

Then

- If $L < 1$, then $\sum_{k}^{\infty} b_k$ converges.
- If $L > 1$, then $\sum_{k}^{\infty} b_k$ diverges.
- If $L = 1$, then you must use another convergence test. The series doesn't compare to a geometric series.

Exercises to submit

1. Let's start easy: suppose that you're asked to consider the convergence of a geometric series:

$\sum_{k=0}^{\infty} a\, r^k.$

- *a*. Try the root test

$\lim_{k\to\infty} |a r^k|^{1/k} = \lim_{k\to\infty} |a|^{1/k} |r| = |r| \lim_{k\to\infty} |a|^{1/k}$

$= |r|\, a^{\lim_{k\to\infty} 1/k}$

$= |r|^0$

In[124]:=
```
b[k_] := a r^k
Limit[Abs[b[k]]^(1/k), k → Infinity, Assumptions → r ∈ Reals]
```

$= |r|$

Out[125]= Abs[r]

- *b*. Try the ratio test

In[126]:=
```
b[k_] := a r^k
Limit[Abs[b[k+1]/b[k]], k → Infinity]
```

$\lim_{k\to\infty}\left|\frac{ar^{k+1}}{ar^k}\right| = \lim_{k\to\infty}|r| = |r|$

Out[127]= Abs[r]

- c. State your conclusion. (What about the cases “on the boundary”?)

The series will converge if |r|<1; it diverges if r=1 or r=-1 (unless a=0...:)

- d. Was one test easier than the other?

The ratio test was easier. Way easier!

2. Still easy, but maybe for the wrong reason: suppose that you’re asked to consider the convergence of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$. We know that this is conditionally convergent (by the AST), but not absolutely convergent (since the harmonic series diverges). What do the root and ratio tests tell us?

- *a*. Try the root test (you need to actually calculate the limit).

$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{k}\right|^{1/k}$

In[128]:=
```
b[k_] := 1/k
Limit[Abs[b[k]]^(1/k), k → Infinity, Assumptions → r ∈ Reals]
```

Out[129]= 1

$= \lim_{k\to\infty}\left(\frac{1}{k}\right)^{1/k} \quad \frac{1}{k}\ln\left(\frac{1}{k}\right)$

$= \lim_{k\to\infty} e^{\frac{-\ln(k)}{k}} \quad \lim_{k\to\infty}0$

$= e^0 = 1$

- *b*. Try the ratio test (you need to actually calculate the limit).

In[130]:=
```
Limit[Abs[b[k+1]/b[k]], k → Infinity]
```

$\lim_{k\to\infty}\left|\frac{1/(k+1)}{1/k}\right| = \lim_{k\to\infty}\frac{k}{k+1} = 1$

Out[130]= 1

- c. What can you conclude from these tests?

We can’t conclude anything!

- d. Was one test easier than the other?

The ratio test was easier.

3. Still easy, but again for the wrong reason: suppose that you’re asked to consider the convergence of the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. We know that this is absolutely convergent (as a positive p-series). What do the root and ratio tests tell us?

- *a*. Try the root test (you need to actually calculate the limit).

$\lim_{k\to\infty}\left|\frac{1}{k^2}\right|^{1/k} = \lim_{k\to\infty}\left(\frac{1}{k^2}\right)^{1/k} = \quad \frac{1}{k}\ln\left(\frac{1}{k^2}\right) \quad \frac{1}{k}(-2\ln k)$

In[131]:=
```
b[k_] := 1/k^2
Limit[Abs[b[k]]^(1/k), k → Infinity, Assumptions → r ∈ Reals]
```

Out[132]= 1

$\lim_{k\to\infty} e \ldots = \lim_{k\to\infty} e^{\ldots} = e^0 = 1$

- *b*. Try the ratio test (you need to actually calculate the limit).

$\lim_{k\to\infty}\left|\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}}\right| = \lim_{k\to\infty}\frac{k^2}{(k+1)^2} = 1$

In[133]:=
```
Limit[Abs[b[k + 1] / b[k]], k → Infinity]
```

Out[133]= 1

- c. What can you conclude from these tests?

We can’t conclude anything!

- d. Was one test easier than the other?

The ratio test was easier.

4. As we will soon see there are series representation for many well-known functions. However, if the series do not converge, the series cannot represent the function.

- a. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, for any x where the series converges.
  - Determine if this series converges or not if $x = -1$. If it does use a partial sum, along with an appropriate error estimate, to approximate e^{-1} with error less than 0.0001.

In[223]:=
```
(* In the case of x=-1, we have an alternating series,
which converges by the AST. We need the first neglected term to be < 0.0001. *)
TableForm[Table[{k, 1 / (k + 1) ! - 0.0001}, {k, 0, 10}]]
enn = 7;
partial = N[Sum[(-1) ^k / k !, {k, 0, enn}]];
true = N[E ^ (-1)];
error = Abs[partial - true];
TableForm[
 Transpose[{{true, partial, enn, error}}]
 , TableHeadings → {{"True", "Partial", "No. of terms", "Abs. Error"}}
]
```

Out[223]//TableForm=

0	0.9999
1	0.4999
2	0.166567
3	0.0415667
4	0.00823333
5	0.00128889
6	0.0000984127
7	-0.0000751984
8	-0.0000972443
9	-0.0000997244
10	-0.0000999749

$x = -1$

$k = 7$ is the 1st value for which $\frac{1}{(k+1)!} - .0001 < 0$

i.e., $.0001 > \frac{1}{(k+1)!}$

Out[223]//TableForm=

True	0.367879
Partial	0.367857
No. of terms	7
Abs. Error	0.0000222983

  - Determine if this series converges or not if $x = -2$. If it does use a partial sum, along with an appropriate error estimate, to approximate e^{-2} with error less than 0.0001.

In[217]:=
```
(* In the case of x=-2, we have an alternating series,
which converges by the AST. We need the first neglected term to be < 0.0001. *)
TableForm[Table[{k, 2^(k+1)/(k+1)! - 0.0001}, {k, 0, 10}]]
enn = 10;
partial = N[Sum[(-2)^k/k!, {k, 0, enn}]];
true = N[E^(-2)];
error = Abs[partial - true];

TableForm[
 Transpose[{{true, partial, enn, error}}]
 , TableHeadings → {{"True", "Partial", "No. of terms", "Abs. Error"}}
]
```

Out[217]//TableForm=

0	1.9999
1	1.9999
2	1.33323
3	0.666567
4	0.266567
5	0.0887889
6	0.0252968
7	0.00624921
8	0.00131093
9	0.000182187
10	-0.0000486933

$x=-2$

$k=10$ is the 1st value for which $\frac{2^{k+1}}{(k+1)!} < .0001$

Out[222]//TableForm=

True	0.135335
Partial	0.135379
No. of terms	10
Abs. Error	0.0000439055

- *b.* $\ln(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$, for any x where the series converges.
  - Determine if this series converges or not if $x = 0.5$. If it does, use a partial sum, chosen using an appropriate error estimate, to approximate ln(0.5) with error less than 0.0001.

If $x = \frac{1}{2}$, we hope that

$$\ln\left(\tfrac{1}{2}\right) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left(\frac{1}{2}-1\right)^k}{k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left(-\frac{1}{2}\right)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)}{2^k \cdot k}$$

$$= \sum_{k=1}^{\infty} (-1)\left(\tfrac{1}{2}\right)^k \cdot \tfrac{1}{k}$$

Converges by Absolute convergence test

$\left|(-1)\left(\frac{1}{2}\right)^k \frac{1}{k}\right| < \left(\frac{1}{2}\right)^k$

$\sum_{k=1} \left(\frac{1}{2}\right)^k$ geometric, $|r| < 1$,

In[203]:=
```
(* We can choose the comparison with the geometric series (1/2)^k,
and choose k such that the remainder terms of that series is < 0.0001.  *)
Clear[n]
soln = Solve[(1/2)^(n+1)/(1-1/2) == 0.0001, n];
enn = n /. soln[[1]];
enn = Ceiling[enn]
partial = N[Sum[((-1)^(k+1) (-1/2)^k)/k, {k, 1, enn}]];
true = N[Log[0.5]];
error = Abs[partial - true];

TableForm[
 Transpose[{{true, partial, enn, error}}]
 , TableHeadings → {{"True", "Partial", "No. of terms", "Abs. Error"}}
]
```

$\left(\frac{1}{2}\right)^n = 0.0001$

$n = \text{ceiling}\left(\frac{\ln(0.0001)}{-\ln(2)}\right)$

$x = .5$

Out[206]= 14

Out[210]//TableForm=

True	-0.693147
Partial	-0.693143
No. of terms	14
Abs. Error	3.84047×10^{-6}

- Determine if this series converges or not if $x = 2$. If it does, use a partial sum, chosen using an appropriate error estimate, to approximate ln(2) with error less than 0.0001.

In[191]:=
```
(* In the case of x=2, we have an alternating series,
which converges by the AST. We need the first neglected term to be < 0.0001. *)
(* 1/(k+1)=1/10000  when k=9999, so take 10000 terms: *)
partial = N[Sum[(-1)^(k+1)/k, {k, 1, 10000}]];
true = N[Log[2]];
error = Abs[partial - true];

TableForm[
 Transpose[{{true, partial, error}}]
 , TableHeadings → {{"True", "Partial", "Abs. Error"}}
]
```

A

$x = 2$ ✓

Out[194]//TableForm=

True	0.693147
Partial	0.693097
Abs. Error	0.0000499975