Lab 13 Notes

We start with Taylor Polynomials, $f(x)$ & a center $x=a$.

$$f(x) \approx f(a)$$ (bad approx, but gets f right at a)

$$\simeq f(a) + f'(a)(x-a)$$ (better!)

$$\simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$ (better yet!)

$$\vdots$$

n^{th} degree Taylor polynomial

$$\simeq \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$ (pretty darned good)

$$\text{“}\simeq\text{”} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$ Taylor series

Let's consider a particular example:

$$f(x) = \ln(1-x)$$

center of $a = 0$

$$f(x) \approx f(a) = f(0) = 0$$

(constant - Taylor polynomial)

To do better, compute derivatives:

$$f'(x) = -\frac{1}{1-x}$$

$$f(x) \approx 0 + f'(a)(x-a)$$

$$\approx 0 - 1 \cdot (x) = -x$$

$$f''(x) = -\frac{1}{(1-x)^2}$$

$$f'''(x) = -\frac{2!}{(1-x)^3}$$

$$\boxed{f^{(4)}(x) = -\frac{3!}{(1-x)^4}}$$

$$\boxed{f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}}$$

$$\boxed{f^{(n)}(0) = -(n-1)!}$$

$$f(x) \approx 0 - x + \frac{(-1)}{2!}(x-0)^2 + \frac{(-2!)}{3!}(x-0)^3$$

$$+ \frac{(-3)!}{4!}(x-0)^4 =$$

$$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$T_4(x)$ - Fourth degree Taylor Poly.

$$f(x) = \ln(1-x) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=1}^{\infty} \frac{-(k-1)!}{k!} x^k = \sum_{k=1}^{\infty} -\frac{x^k}{k}$$

$$\boxed{f(x) = \sum_{k=1}^{\infty} -\frac{x^k}{k}}$$

for some values of x

exactly what you get when you differentiate this series!

$$\frac{-1}{1-x} = f'(x) = -\frac{1}{1-x}$$

$$= -\sum_{k=0}^{\infty} x^k$$

Holds for $|x|<1$

Interval of convergence of the Taylor Series for $f(x)$,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot x \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot |x| = |x|$$

For convergence, demand

$$|x| < 1$$

$$[-1, 1)$$?

Interval of Convergence

What happens on the boundary?

$\ln(1-x)$ blows up at $x=1$

$$\ln(1-(-1)) = \ln(2)$$

$$-\left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\right)$$

Alternating series harmonic

Suppose we want to use $T_4(x)$ to approximate $f(x)$ on $[-\frac{1}{2}, \frac{1}{2}]$

$$f(x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

How bad could the error be?

$$R_4(x) = \sum_{k=5}^{\infty} \frac{-x^k}{k}$$ (all the rest of the Taylor series)

(The remainder term)

$$|R_4(x)| \leq \frac{M}{5!} \cdot |x|^5$$ where

M is a bound on the size of the 5th derivative (1st neglected term) on the interval $[-\frac{1}{2}, \frac{1}{2}]$

$$f^{(5)}(x) = \frac{-4!}{(1-x)^5}$$

where is this biggest on this interval?

$$M \geq \frac{4!}{(1-\frac{1}{2})^5} = \frac{4!}{\frac{1}{2}^5} = 32 \cdot 4!$$

Let $M = 32 \cdot 4!$

$$\therefore \quad |R_4(x)| \le \frac{32 \cdot 4!}{5!} \cdot |x|^5$$

$$\le \frac{32}{5} \cdot |x|^5 \le \frac{32}{5} \cdot \left(\frac{1}{2}\right)^5$$

on this interval.

$|R_4(x)| \le 1/5$ on this interval.