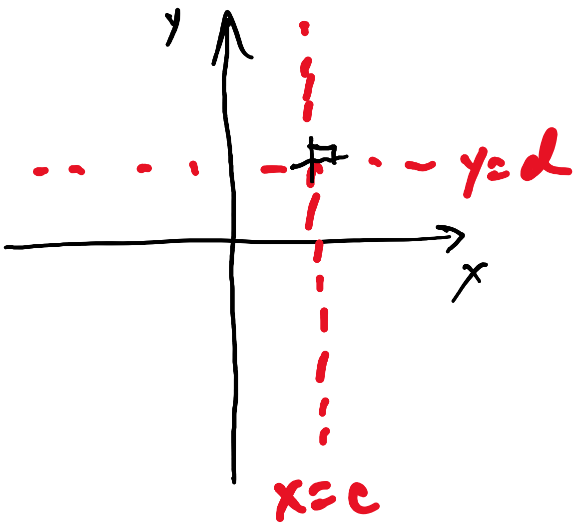
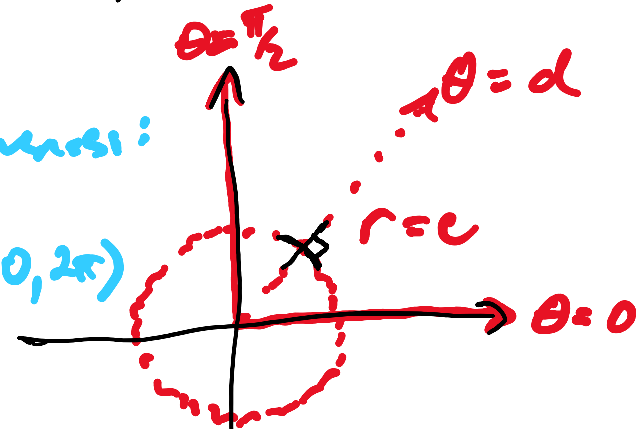
# Lab 15 notes:

## Cartesian versus Polar



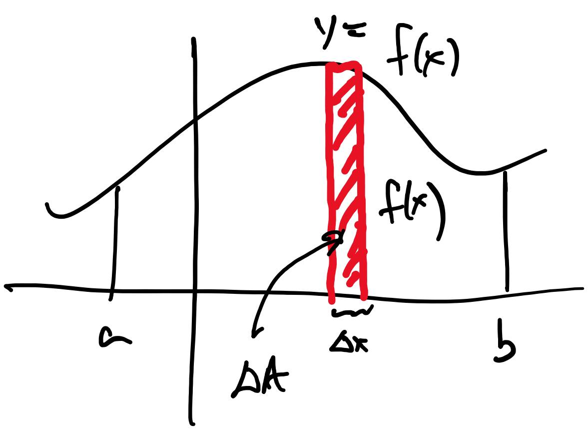

Vertical + horizontal lines

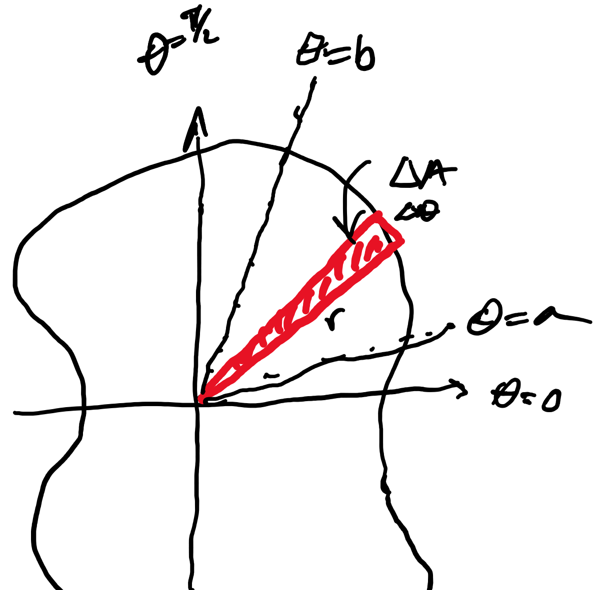
For uniqueness:
$r \geq 0$
$\theta \in [0, 2\pi)$




Polar coordinates are great for circles centered at the origin, + rays eminating from the origin.

## Areas




$$\Delta A = \Delta x\, f(x)$$

in the limit $\Delta x \to 0$

$$dA = dx\, f(x)$$

$A = ?$

$$\Delta A \approx \frac{1}{2} \cdot r \cdot r\Delta\theta$$
$$= \frac{1}{2} r^2 \Delta\theta$$

$$\boxed{A=\int_a^b dx\, f(x)}$$

$$\downarrow \; dA = \frac{1}{2} r^2 d\theta$$

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

$$\boxed{A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta}$$

Example:

$r = \overbrace{1-\cos\theta}^{f(\theta)}$ (cardiod)


$$A = \int_0^{2\pi} dA = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 - \underbrace{\cancelto{0}{2\cos\theta}}_{\text{integrating cosine over a period}} + \cos^2(\theta)\right) d\theta$$

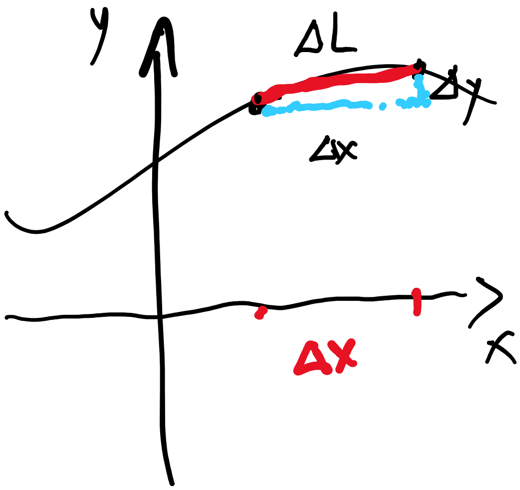
$$= \frac{1}{2}\left[\int_0^{2\pi} d\theta + \int_0^{2\pi} d\theta \cos^2(\theta)\right]$$

$$= \frac{1}{2}\left[2\pi + \int_0^{2\pi} d\theta\left[\frac{1}{2} + \cancelto{0}{\frac{1}{2}\cos 2\theta}\right]\right]$$

$$= \frac{1}{2}\left[2\pi + \frac{1}{2}\int_0^{2\pi} d\theta\right]$$

$$= \frac{1}{2}\left[3\pi\right] = \boxed{\frac{3}{2}\pi} \approx 4.7$$

# Length




$$\Delta L = \sqrt{\Delta x^2 + \Delta y^2}$$

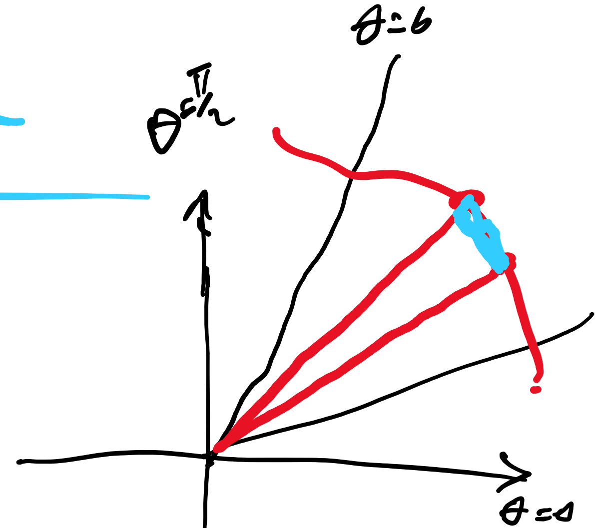
$$= \sqrt{\Delta x^2 + \frac{\Delta x^2}{\Delta x^2}\cdot\Delta y^2}$$

$$= \Delta x\sqrt{1 + \frac{\Delta y^2}{\Delta x^2}}$$

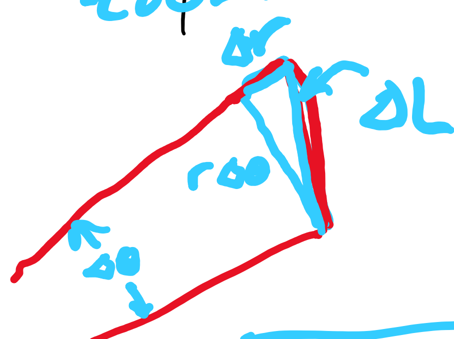
$$= \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$dL = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$L = \int_a^b dL = \int_a^b dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$




Zoom in:




$$\Delta L = \sqrt{\Delta r^2 + (r\Delta\theta)^2}$$

$$= \sqrt{\frac{\Delta\theta^2}{\Delta\theta^2}\Delta r^2 + r^2\Delta\theta^2}$$

$$= \Delta\theta\sqrt{\frac{\Delta r^2}{\Delta\theta^2} + r^2}$$

$$dL = d\theta\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

Example: What's the length of the heart for $r = 1 - \cos\theta$

$$L = \int_0^{2\pi} dL = \int_0^{2\pi} d\theta \sqrt{(\sin\theta)^2 + (1-\cos\theta)^2}$$

$$= \int_0^{2\pi} d\theta \sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}$$

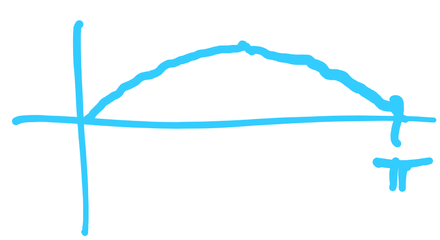
$$= \int_0^{2\pi} d\theta \sqrt{2 - 2\cos\theta}$$

$$= \int_0^{2\pi} d\theta \sqrt{4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos\theta\right)}$$

$$= \int_0^{2\pi} d\theta \, 2\sqrt{\tfrac{1}{2} - \tfrac{1}{2}\cos\theta}$$

$$= \int_0^{2\pi} d\theta \, 2\sqrt{\sin^2(\theta/2)}$$

$$= \int_0^{2\pi} d\theta \, 2 \cdot \left|\sin(\theta/2)\right|$$



$$= \int_0^{2\pi} d\theta \, 2 \sin\frac{\theta}{2} = 2\left(-2\cos\theta/2\right)\Big|_0^{2\pi}$$

$$= -4(-1-1) = \boxed{8}$$

$$L = \int_a^b d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$