Lab 5 - Integration by Parts & Trig integrals.

Consider

$$\int \underline{x^3}\, \underline{\ln(x)}\, dx$$

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$\int [f(x)g(x)]'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$I = \int \underline{f'(x)g(x)}\,dx = f(x)g(x) - \int \underline{f(x)g'(x)\,dx}$$

$x^3 \quad \ln(x)$

$$\boxed{\begin{aligned} f(x) &= \tfrac{1}{4}x^4 \\ g'(x) &= \tfrac{1}{x} \end{aligned}}$$

$$= \frac{1}{4}x^4 \cdot \ln(x) - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{4}x^4 \ln(x) - \int \frac{1}{4}x^3\,dx$$

Consider

$$\underline{\underline{I}} = \int e^{2x} \cos(x)\,dx$$

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$g'(x) = \cos(x)$

$g(x) = \sin(x)$

$$= \int f(x)\, g'(x)\, dx$$

$f(x) = 2e^{2x}$
$f'(x) = 4e^{2x}$
$g'(x) = \sin(x)$
$g(x) = -\cos(x)$

$$= e^{2x}\sin(x) - \int \underbrace{2e^{2x}}_{f(x)} \underbrace{\sin(x)\, dx}_{g'(x)\, dx}$$

$$I = e^{2x}\sin(x) - \left[2e^{2x}(-\cos(x)) - \int 4e^{2x}(-\cos(x))\, dx \right]$$

$$= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\underbrace{\int e^{2x}\cos(x)\, dx}_{I}$$

$$5I = e^{2x}(\sin(x) + 2\cos(x))$$

$$I = \frac{1}{5}e^{2x}(\sin(x) + 2\cos(x)) + C$$

$$I = \int \sin^2(x)\cos^5(x)\, dx$$

$$= \int \underbrace{\sin^2(x)\cos^4(x)}_{\substack{\text{write this in} \\ \text{terms of} \\ u=\sin(x)}} \underbrace{\cos(x)\, dx}_{du}$$

$\cos^2(x) + \sin^2(x) = 1$

$\cos^2(x) = 1 - \sin^2(x)$

$$= \int \sin^2(x)\left(\cos^2(x)\right)^2 \cos(x)\, dx$$

$$= \int \sin^2(x)\left(1 - \sin^2(x)\right)^2 \cos(x)\, dx$$

$$= \int u^2 (1-u^2)^2 \, du$$

(don't forget - rewrite your answer in terms of x)

$$I = \int \cos^4(x) \, dx$$

$$= \int \left(\cos^2(x)\right)^2 dx$$

$$I = \int \left(\tfrac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \frac{1}{4} \int \left(1 + 2\cos(2x) + \cos^2(2x)\right) dx$$

$$= \frac{1}{4} \int \left(1 + 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x))\right) dx$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$\cos(x+x) = \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\cos(2x) = \cos^2(x) - (1-\cos^2(x))$$

$$= 2\cos^2(x) - 1$$

$$\boxed{\cos^2(x) = \tfrac{1}{2}(1+\cos(2x))}$$

$$I = \int \sec^4(x)\tan^5(x)\,dx$$

$$= \int \frac{1}{\cos^4(x)} \cdot \frac{\sin^5(x)}{\cos^5(x)}\,dx$$

$$= \int \frac{\sin^4(x) \cdot \sin(x)\,dx}{\cos^9(x)}$$

$u = \cos(x)$

$du = -\sin(x)\,\cancel{dx}$

$$= \int \frac{(\sin^2(x))^2 \sin(x)\,dx}{\cos^9(x)}$$

$$= \int \frac{(1 - \cos^2(x))^2 \sin(x)\,dx}{\cos^9(x)}$$

$$= \int \frac{(1-u^2)^2}{u^9}(-du)$$

one way....

$$I = \int \sec^4(x)\tan^5(x)\,dx$$

$$\boxed{\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}}$$

$1 + \tan^2(x) = \sec^2(x)$

$$= \int \sec^2(x)\tan(x)\,\sec^2(x)\,dx$$

$$= \int \sec^2(x)\tan(x)\left(\frac{d}{dx}\tan(x)\right)dx$$

$$= \int (1+\tan^2(x))\tan(x)\frac{d}{dx}(\tan(x))\,dx$$

$$= \int (1+u^2)\,u\,du$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$I = \int \sec^4(x)\tan^5(x)\,dx$$

$$= \int \sec^3(x)\tan^4(x)\underbrace{\sec(x)\tan(x)\,dx}_{du \text{ if } u=\sec(x)}$$

$$= \int \sec^3(x)\left(\sec^2(x)-1\right)^2\frac{d}{dx}(\sec(x))\,dx$$

$$= \int u^3(u^2-1)^2\,du$$

$$I = \int \sqrt{4-x^2}\,dx$$

$$\boxed{\begin{array}{l} x = 2\sin(t) \\ dx = 2\cos(t)\,dt \end{array}}$$

$$= \int \sqrt{4 - 4\sin^2(t)}\, 2\cos(t)\, dt$$

$$= \int 2\sqrt{1 - \sin^2(t)}\, 2\cos(t)\, dt$$

$$= \int 2\sqrt{\cos^2(t)}\, 2\cos(t)\, dt$$

$$= \int 2\,|\cos(t)|\, 2\cos(t)\, dt$$

$$= 4\int \cos^2(t)\, dt$$

$$= 4\int \frac{1}{2}\left(1 + \cos(2t)\right) dt$$

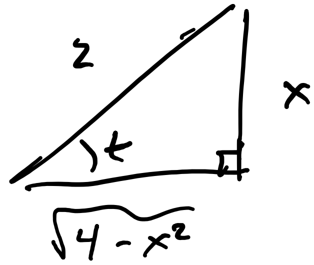
$$= 2\int \left(1 + \cos(2t)\right) dt$$

$$I = 2\left(t + \frac{1}{2}\sin(2t)\right) + C$$

How do I write this in terms of x?

$$x = 2\sin(t) \iff \frac{x}{2} = \sin(t)$$

$$t = \arcsin\left(\frac{x}{2}\right)$$



$$I = 2\left(\arcsin\left(\tfrac{x}{2}\right) + \tfrac{1}{2}\left(2\sin(t)\cos(t)\right)\right) + C$$

$$= 2\left(\arcsin\left(\tfrac{x}{2}\right) + \tfrac{1}{2}\left(x\,\frac{\sqrt{4-x^2}}{2}\right)\right) + C$$

$$\boxed{\begin{aligned}\sin(2t) &= \sin(t+t)\\ &= 2\sin(t)\cos(t)\end{aligned}}$$

$$I = 2\arcsin\left(\tfrac{x}{2}\right) + x\sqrt{4-x^2}/2 + C$$