Lab 8: Student Assignment

Week 8

MAT 229, Spring 2021

Exercises to submit

These must be done by hand on paper (or typed into Mathematica). Use Mathematica to check your work. Submit as pdf files in Canvas.

Exercise 1

Consider the improper integral $\int_0^\infty x\, e^{-x}\, dx = \lim_{R\to\infty} \int_0^R x\, e^{-x}\, dx$.

a. Use an appropriate technique of integration to find an antiderivative.

Integration by parts: differentiate f(x)=x; antidifferentiate g'(x)=e^{-x}:

$\lim_{R\to\infty}(-xe^{-x}\,|_0^R + \lim_{R\to\infty}\int_0^R e^{-x}\, dx = \lim_{R\to\infty}(-xe^{-x}\,|_0^R - e^{-x}\,|_0^R)$

b. The limit is an indeterminate limit. What form is it? $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, 0^∞, 1^∞, $\infty-\infty$

$\frac{\infty}{\infty}$

$\lim_{R\to\infty} -xe^{-x} = \lim_{R\to\infty} \frac{-x}{e^x} \left(\frac{\infty}{\infty}\right)$

c. Compute the limit using appropriate techniques.

$\lim_{R\to\infty} \frac{-1}{e^x} = 0$ (l'Hopital)

```
In[4]:= Integrate[x E^(-x), {x, 0, Infinity}]
        Limit[-R E^(-R) - E^(-R), R → Infinity] + 1
Out[4]= 1
Out[5]= 1
```

Exercise 2

Let $A = \int_0^1 \frac{1}{\sqrt{x} + |\sin(x)|}\, dx$

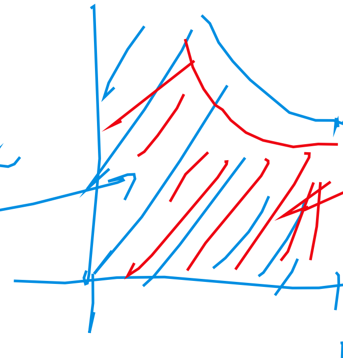
a. How does $\frac{1}{\sqrt{x} + |\sin(x)|}$ compare with $\frac{1}{\sqrt{x}}$?

$\frac{1}{\sqrt{x} + |\sin(x)|} < \frac{1}{\sqrt{x}}$ on (0,1]; they both asymptote to infinity at x=0.

b. Does $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converge or diverge? If it converges, what does it converge to?

In[6]:=
```
Integrate[1 / Sqrt[x], {x, 0, 1}]
```
Out[6]= 2

finite



A finite! (less area)

$$\lim_{r\to 0} 2\sqrt{x}\,\Big|_r^1 = 2$$

c. What conclusions can you draw about whether A converges or diverges, if any. If it converges, approximate it with the midpoint rule using $n = 10$.

Because the denominator becomes bigger with the term $|\sin(x)|$, the quotient is smaller; and so, since the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent, so is A.

In[48]:=
```
a = 0.0;
b = 1.0;
f[x_] := 1 / (Sqrt[x] + Abs[Sin[x]])
n = 10
dx = (b - a) / n
mid = dx * Sum[f[a + (k - 0.5) dx], {k, 1, n}]
NIntegrate[f[x], {x, a, b}]
```
Out[51]= 10

Out[52]= 0.1

Out[53]= 1.21331

Out[54]= 1.40291

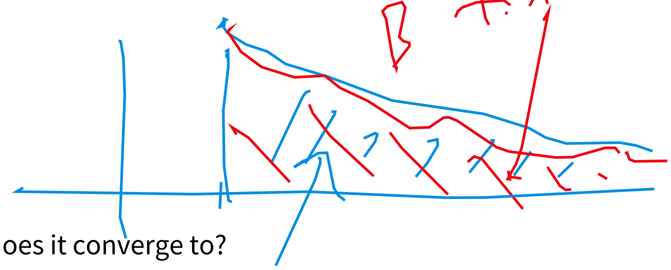
so we can estimate A w/ midpoint

Mathematica says...

Exercise 3

Let $B = \int_1^\infty \frac{3}{x^2 + |\sin(x)|}\,dx$

B finite!



a. How does $\frac{3}{x^2 + |\sin(x)|}$ compare with $\frac{3}{x^2}$?

$\frac{3}{x^2 + |\sin(x)|} < \frac{3}{x^2}$?

b. Does $\int_1^\infty \frac{3}{x^2}\,dx$ converge or diverge? If it converges, what does it converge to?

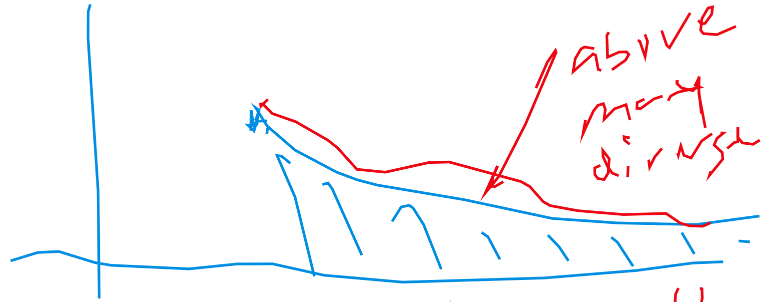
finite

In[55]:=
```
Integrate[3 / x^2, {x, 1, Infinity}]
```
Out[55]= 3

c. What conclusions can you draw about whether B converges or diverges, if any. If it converges, what can you say about the value of B?

Well, it's certainly less than 3.

In[60]:=
```
a = 1.0;
b = Infinity;
f[x_] := 1 / (x^2 + Abs[Sin[x]])
NIntegrate[f[x], {x, a, b}]
```
Out[63]= 0.805535



above - may diverge!

Exercise 4

Let $C = \int_1^\infty \frac{1}{x^2 - |\sin(x)|}\, dx$

a. How does $\frac{1}{x^2 - |\sin(x)|}$ compare with $\frac{1}{x^2}$?

$\frac{1}{x^2 - |\sin(x)|} > \frac{1}{x^2}$?

b. Does $\int_1^\infty \frac{1}{x^2}\, dx$ converge or diverge? If it converges, what does it converge to?

From above (same integral multiplied by 3), it must converge to 1.

c. What conclusions can you draw about whether C converges or diverges, if any.

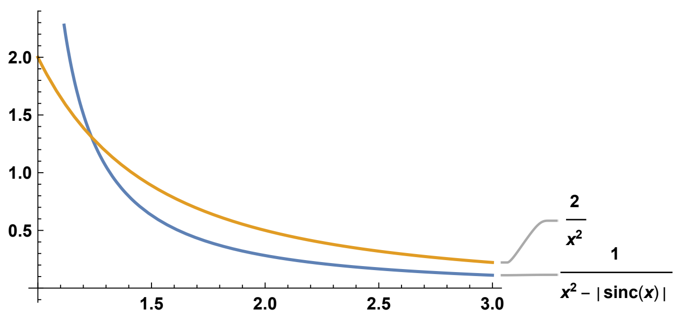
Based on that, we can't draw any conclusions -- since C> $\int_1^\infty \frac{1}{x^2}\, dx$ which is convergent. However....

$\frac{1}{x^2 - |\sin(x)|} < \frac{1}{0.5\, x^2}$ very soon after leaving the gate at x=1:

(it doesn't, but it can't be determined by comparison w/ $1/x^2$)

In[64]:=
```
Plot[{1/(x^2 - Abs[Sinc[x]]), 2/(x^2)}, {x, 1, 3}, PlotLabels → Automatic]
```

Out[64]=



Thus we conclude that it is convergent. However we cannot say much about its value.

Exercise 5:

An application of improper integration is the calculation of the work (energy) done in lifting an object of mass m (in kilograms) from the surface of the earth to a distance D from the center of the earth. To lift it "entirely away" from Earth's influence, you need to let $D \to \infty$, making the integral improper.

Work is force times the distance over which the force acts. The force f due to the Earth's gravity is an "inverse square" law: $f(x) = \frac{m\, G\, M_e}{(R_e + x)^2}$, where M_e is the mass of the Earth, R_e is the radius of the Earth, and G is the universal gravitational constant. (The "inverse square" part is the "$(R_e + x)^2$" in the denominator.) So as we move away from the Earth's surface (at x=0), in little steps of dx, we have to multiply by the force of gravity, which is acting against the movement.

Now you may be familiar with the acceleration due to gravity at the Earth's surface, g, which is roughly 9.8 m/s^2:

$$g_0 = \frac{G\,M_e}{R_e^2} = 9.81998\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$G = 6.67408 \cdot 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}.$$

So we can re-express $f(x)$ as $f(x) = m\,g\,\frac{R_e{}^2}{(R_e + x)^2}$, where the radius of the Earth is 6360 km.

The work done to remove our object of mass m to a distance D from the surface of the Earth is

$$W = \int_0^D dW(x) = \int_0^D f(x)\,dx$$

Here is the command to initialize the important parameters we find in *f(x)*, and the problems (the rest is up to you):

In[73]:=
```
re = 6371 * 1000 (* radius of the Earth, in meters, according to Google!:) *) ;
g = 9.81998
  (* acceleration due to gravity at the surface of the Earth, in m/s^2 *);
spaceStation = 408 000 (* height above the Earth, in meters *);
m = 1 (* in kg *);
```

a. The international space station is at approximately 408 km from the Earth's surface. How much work (in Newton meters, or "Joules") is required to lift an object of one kilogram to the space station? (Just for comparison purposes, it takes 5.93538×10^9 Joules to lift 700 kg to 1,000,000 meters.)

I had mistaken, written pounds in the lab - sorry!

In[80]:=
```
Integrate[m g re^2/(re + x)^2, {x, 0, spaceStation}]
Integrate[700 g re^2/(re + x)^2, {x, 0, 10^6}]
```

Out[80]= 3.76541×10^6

Out[81]= 5.94141×10^9

b. How much work (in Newton meters, or Joules) is required to lift an object of one kilogram to "infinity" (forgetting the Sun, and other planets, and all the other stars, etc. Plus it's a really long trip...:)? Do this by hand, as a limit, and confirm with Mathematica.

In[79]:=
```
Integrate[m g re^2/(re + x)^2, {x, 0, Infinity}]
```

Out[79]= 6.25631×10^7