Lab 8: Instructors' notes

Week 8

MAT 229, Spring 2021

Improper integrals

Improper integrals $\int_a^b f(x)\,dx$ take one of two forms:

- At least one of the limits of integration is infinite.
- The integrand has a vertical asymptote for value of x in interval $[a, b]$.

In either case, turn it into a limit problem.

Examples

- $\int_0^\infty e^{-3x}\,dx$
- $\int_2^\infty \frac{x}{x^2-1}\,dx$
- $\int_0^1 \frac{x}{1-x^2}\,dx$
- $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$

Standards

- $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \le 1 \\ \text{converges} & \text{if } p > 1 \end{cases}$
- $\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \ge 1 \\ \text{converges} & \text{if } p < 1 \end{cases}$

Note that p=1 is the special case that diverges in either domain of integration.

Mathematica and improper integrals

Mathematica can evaluate improper integrals. Let's try to integrate 1/x, from 1 to Infinity:

```
In[ ]:= Integrate[1 / x, {x, 1, Infinity}]
```

```
In[•]:= NIntegrate[1 / x, {x, 1, Infinity}]
```

Mathematica will try to integrate 1/x, from 0 to 1:

```
In[•]:= Integrate[1 / x, {x, 0, 1}]
```

```
In[•]:= NIntegrate[1 / x, {x, 0, 1}]
```

Mathematica can try to evaluate improper integrals. But if the integral diverges, it may provide a value. **Beware**!

Now we've frustrated Mathematica enough: let's give it one it can actually evaluate:

```
In[•]:= Integrate[1 / x^3, {x, 1, Infinity}]
```

```
In[•]:= NIntegrate[1 / x^3, {x, 1, Infinity}]
```

Now we're back to our old tricks:

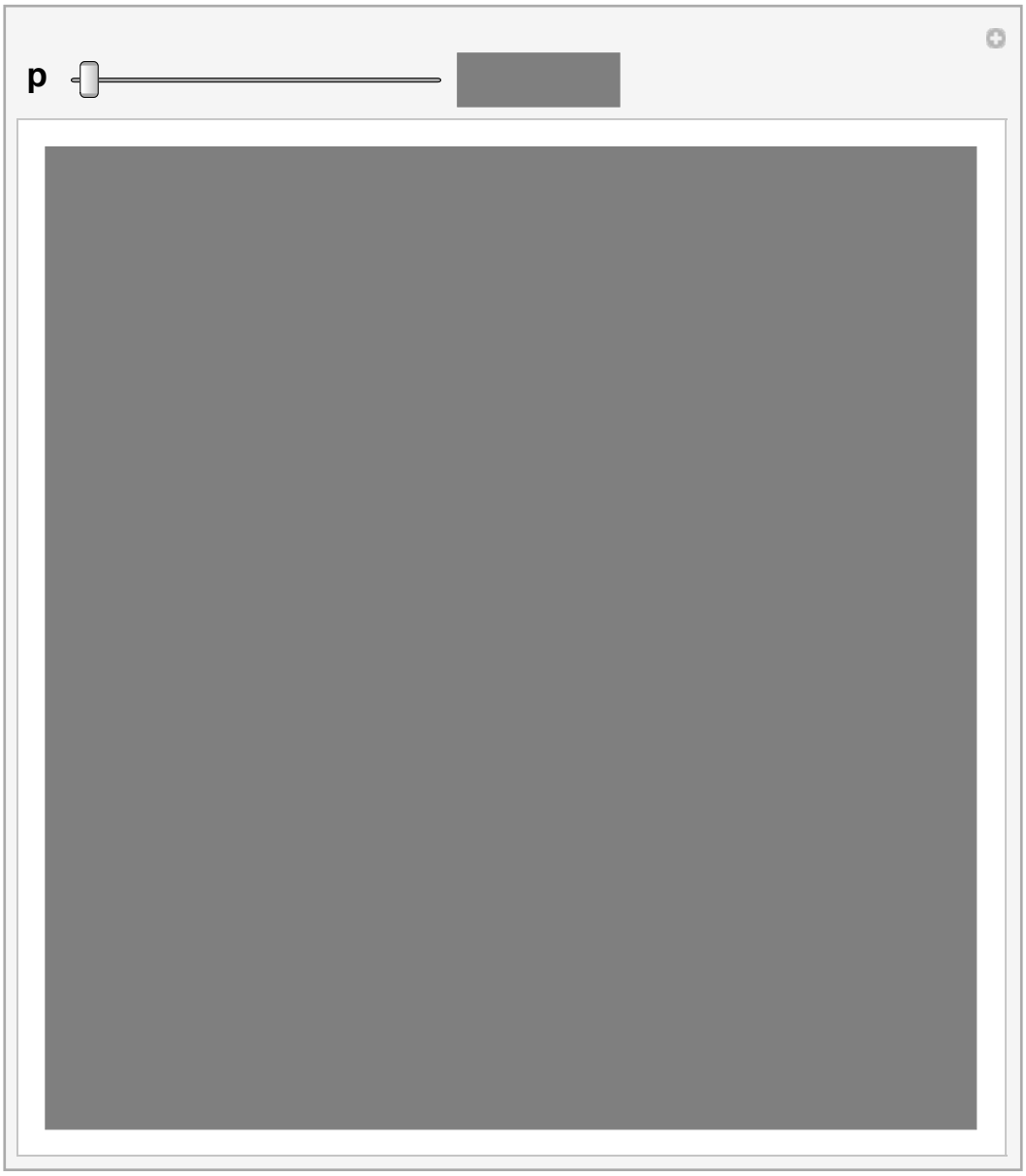
```
In[•]:= Integrate[1 / x^3, {x, 0, 1}]
```

```
In[•]:= NIntegrate[1 / x^3, {x, 0, 1}]
```

The attached animation illustrates the general case for the powers:

- $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \le 1 \\ \text{converges} & \text{if } p > 1 \end{cases}$
- $\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \ge 1 \\ \text{converges} & \text{if } p < 1 \end{cases}$

Out[ ]=



Comparison test

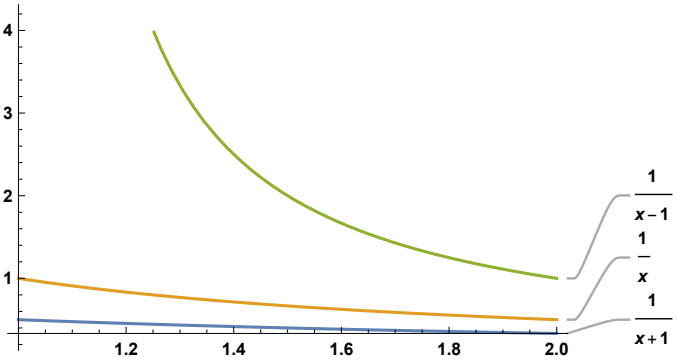
Inequalities

Which one is bigger $\frac{1}{2}$ or $\frac{1}{4}$?

If numerators are the same, a bigger denominator makes a smaller value (x>1):

$$\frac{1}{x+1} < \frac{1}{x} < \frac{1}{x-1}$$

Out[ ]=



Example

Does $\int_1^\infty \frac{1}{x^2}\,dx$ converge or diverge? If so, what is its value?

How does $\frac{1}{x^2+x}$ compare with $\frac{1}{x^2}$? What does that say about its graph?

What has to be true about $\int_1^\infty \frac{1}{x^2+x}\,dx$?

Example

Does $\int_0^1 \frac{1}{2x^2}\,dx$ converge or diverge? If so, what is its value?

How does $\frac{1}{2x^2-x^3}$ compare with $\frac{1}{2x^2}$? What does that say about its graph?

What has to be true about $\int_0^1 \frac{1}{2x^2+x^3}\,dx$?