Build a Brachistochrone and Captivate Your Class

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Introduction

Cliff Long (1931–2002) [5] was a master teacher whose office was a wonderful place to visit, for it was crammed with a wealth of teaching devices. From his early *Bug on a Band* [6], to his slides and flexible model of quadratic surfaces [3,4], to his head of Abraham Lincoln made with a computer-controlled milling machine [1], and his fascination with knots [7], Cliff was always on the lookout for new ways to illustrate mathematical concepts.

As a young faculty member I went to his office whenever I wondered how best to present some topic in class. He had thought long and hard about everything he taught and was full of ideas about how to enhance learning. Cliff was my mentor and I learned an immense amount about teaching from him.

Of all the things in his office, my favorite was his brachistochrone. I borrowed it to use in talks whenever the Bernoullis were mentioned. The brachistochrone problem was my favorite way to end a class on the integral calculus, for it provided a lovely way to review many of the topics we had studied [10]. Shortly before I retired from Bowling Green State University in 1998, Cliff talked to me about an improved design for the brachistochrone and asked for my suggestions. Little did I know that he was making one for me. I was honored.

Parametric equations

When introducing the topic of parametric equations, a good way to proceed is to arrive in the classroom with your brachistochrone under your arm. When your students arrive, take the circular disk and draw a cycloid on the blackboard (see Figure 1). Be sure to tell the students that a cycloid is the curve generated by a point on the circumfrence of a circle that is rolling along a straight line and that the word derives from the Greek word for circle, *kuklos*. Perhaps they will have heard of epicycloids, those circles rolling on

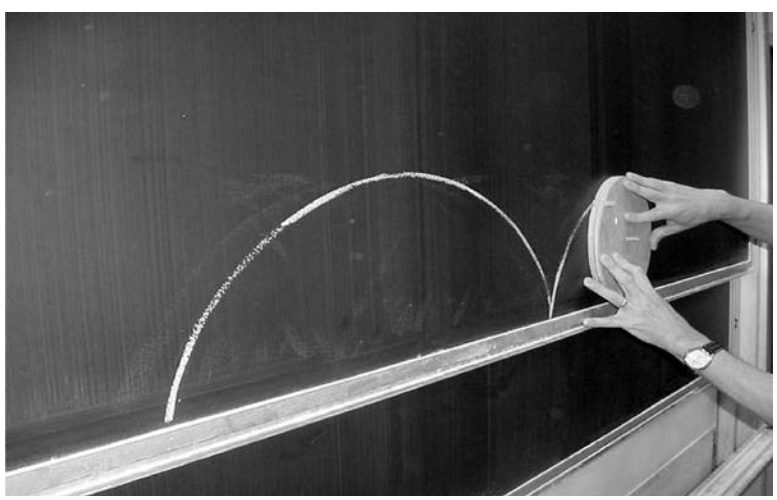

Figure 1. Rolling a curve to produce a cycloid

circles, that are part of Ptolemaic astronomy. Of course you will want to practice drawing a cycloid in advance, for it is a little tricky to get the disk to roll without slipping; also you need to keep enough pressure on the disk—and the chalk—so that the chalk will trace the curve. If the blackboard does not have a protruding edge that you can roll the disk on then you will have to get several students to hold a meter stick tightly against the blackboard so that you will have a firm base to roll the circle on. If the blackboard slides up so that you can work at a more convenient height, that is even better.

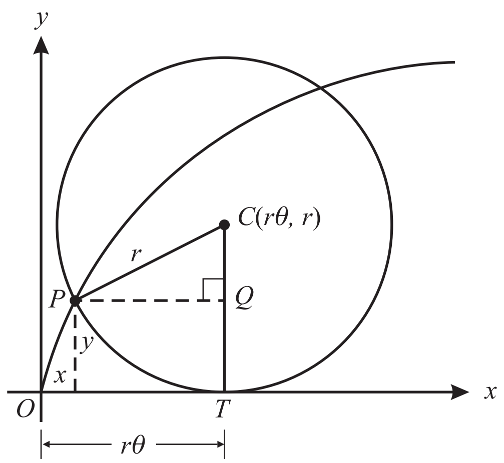

Figure 2. Finding parametric equations for the cycloid

Now we are ready to derive parametric equations for the cycloid. If you place the wooden disk on the base line and then trace around it with chalk, you can draw a circle on the base line of your cycloid, as in Figure 2.

For the parameter we use θ, the angle the circle has rolled through. We begin with $\theta = 0$ and the tracing point P at the origin. If the circle has radius r, then when it rolls through angle θ the circle will roll the distance $OT = r\theta$. To locate the x-coordinate of the point P, first move to the right the distance OT and then left the distance QP, i.e., $r\sin\theta$. Similarly, to obtain the y-coordinate we move up to the center of the circle (TC), i.e., up r. and then down to P, i.e., down CQ or $r\cos\theta$. Thus we have the parametric equations. Moving left (down) accounts for the minus sign in the parametric equations.

$$\begin{cases} x = r\theta - r\sin\theta, \\ y = r - r\cos\theta . \end{cases}$$

At this stage your students should plot these equations on a computer algebra system and observe that the graph they obtain looks like what was sketched on the blackboard by the rolling circle.

While the early history of the cycloid is unknown, Galileo named the curve (it is also called the roulette and trochoid) and attemped to find the area under the curve in 1599. He approximated the area by making a physical model and weighing it. His conclusion was that the area under one arch of the cycloid was about three times the area of the generating circle. One could repeat Galileo's experiment in class by constructing an accurate arch of a cycloid and weighing it—but you will need a chemist to loan you an accurate scale. In 1643, Roberval proved Galileo's conjectured value of 3 was correct. Tangents to the curve were constructed later in the decade by Descartes, Fermat and Roberval, presumably independently. In 1658, Pascal posed a number of problems related to the cycloid. The most interesting of these, the rectification, or arc length, of the curve, was solved by Christopher Wren [17]. In 1686, Leibniz found a Cartesian equation for the curve:

$$y = \sqrt{2x - xx} + \int \frac{dx}{\sqrt{2x - xx}}.$$

Bernoulli's New Year's Day problem

Just nine months after Newton left the Lucasian Professorship at Cambridge to take up "y^{e} Kings business" at the Mint in London, he received a letter from France containing a flysheet printed at Groningen which was dated January 1, 1697. He received it on January 29, 1696; this is no typo, England was not yet on the Gregorian calendar [17]. It was addressed

> To the sharpest mathematicians now flourishing throughout the world, greetings from Johann Bernoulli, Professor of Mathematics.[11]

You might wonder why Johann Bernoulli was teaching in the Netherlands. This was because his older brother, Jacob, held the professorship of mathematics at the university in their native town of Basel; the newly married Johann was forced to look elsewhere. Through the help of Liebniz and L'Hospital he obtained a position at the university in Groningen. Bernoulli's stated aim in proposing this problem sounds admirable:

> We are well assured that there is scarcely anything more calculated to rouse noble minds to attempt work conductive to the increase of knowledge than the setting of problems at once difficult and useful, by the solving of which they may attain to personal fame as it were by a specially unique way, and raise for themselves enduring monuments with posterity. For this reason, I … propose to the most eminent analysts of this age, some problem, by means of which, as though by a touchstone, they might test their own methods, apply their powers, and share with me anything they discovered, in order that each might thereupon receive his due meed of credit when I publically announced the fact. [11]

Bernoulli's new year's present to the mathematical world was a great gift, a difficult problem that would enrich the field:

> To determine the curved line joining two given points, situated at different distances from the horizontal and not in the same vertical line, along which a mobile body, running down by its own weight and starting to move from the upper point, will descend most quickly to the lowest point. [11]

This is the brachistochrone problem. The word was coined by Johann Bernoulli from the Greek words `brachistos' meaning shortest and 'chronos' meaning time. But the problem was not new. In 1638, Galileo attacked it in his last work, *Discorsi e dimostrazioni matematiche, intorno à due nuoue scienze* [*Discourses and Mathematical Demonstrations Concerning Two New Sciences*], but he was unable to solve it. Galileo was only able to prove that a circular arc provided a quicker descent than a straight line. Bernoulli noted this in his flysheet when he wrote that the solution to the brachistochrone problem was not a straight line, but a curve well known to geometers.

Earlier, in June 1696, Johann Bernoulli published a paper in Germany's first scientific periodical, the *Acta eruditorum*, wherein he attempted to show that the calculus was necessary and sufficient to fill the gaps in classical geometry. At the end of the paper the brachistochrone problem was posed as a challenge, setting a deadline in six months, but Bernoulli received no correct solutions. He had received a letter from Leibniz sharing that "The problem attacked me like the apple did Eve in Paradise'' [14] and that he had solved it in one evening. In fact, he had only found the differential equation describing the curve, but had not recognized the curve as an inverted cycloid. Bernoulli and Leibniz interpreted Newton's six month silence to mean the problem had baffled him—indeed he had not seen it. To demonstrate the superiority of their methods, Leibniz suggested the deadline be extended to Easter and that the problem be distributed more widely. So Bernoulli added a second problem, had a flysheet published, and made sure it circulated widely.

The brachistochrone problem was a difficult one. Pierre Varignon and the Marquis de L'Hospital, in France, and John Wallis and David Gregory, in England, were all stumped. But Newton was not. Thirty years later Newton's niece Catherine Barton Conduitt recalled,

> When the problem in 1697 was sent by Bernoulli—Sr. I. N. was in the midst of the hurry of the great recoinage [and] did not come home till four from the Tower very much tired, but did not sleep till he had solved it wch was by 4 in the morning. [15, pp. 582–3; 16, pp. 72–73]

The next day Newton sent his solution to his old Cambridge friend Charles Montague, who was then President of the Royal Society. He published Newton's work anonymously in the February issue of the

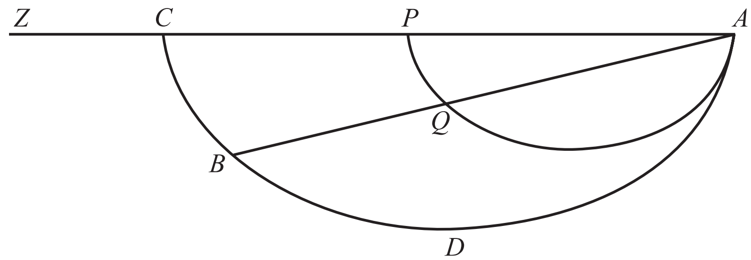

Figure 3. Newton's Construction

Transactions of the Royal Society of London. The trap that Bernoulli and Leibniz had set for Newton failed to snare its game.

Newton provided no justification for his solution, only showing how to construct the necessary cycloid [11, p. 22; 16, p. 75]. He simply drew an arbitrary inverted cycloid with its starting point at A, the higher of the two points, and then drew a line between the two given points, A and B. This line intersects the arbitrary cycloid at Q. Then he found the ratio of the line segment from the starting point to the final point to the line segment from the starting point to the initial cycloid, i.e., AB/AQ. He used this ratio to expand the radius of the initial circle to produce the circle which would generate the desired cycloid.

Derek T. Whiteside, who has published the extremely valuable edition of *The Mathematical Papers of Isaac Newton*, claims that the fact that it took Newton twelve hours to solve these problems indicates that his mathematics was rusty from nine months disuse. It also shows that the gradual decline in Newton's mathematical ability had set in. However, his solution of the brachistochrone problem is counterevidence to the myth that Newton's old age was mathematically barren [16, pp. xii, 3] .

Immediately on receiving the solution of the anonymous Englishman via Basnage de Beauval, Bernoulli wrote Leibniz that he was "firmly confident" that the author was Newton. Leibniz was more cautious on the authorship of the problem, admitting only that the solution was suspiciously Newtonian. Several months later Bernoulli wrote de Beauval that "we know indubitably that the author is the celebrated Mr. Newton; and, besides, it were enough to understand so by this sample, ex ungue Leonem." Within a few weeks this shrewd guess was common knowledge across Europe. The phrase goes back to Plutarch and Lucian, who allude to the ability of the sculptor Phidias to determine the size of a lion given only its severed paw.

Not having succeeded in trapping Newton, Bernoulli lost interest, leaving it to Leibniz to publish the solutions in the May 1697, *Acta* (pp. 201–224). These included Johann's own solution, one by his older brother Jakob, one by L'Hospital (probably produced with the help of Johann Bernoulli), one by Tschirnhaus, and a reprint—seven lines in all—of Newton's. This time Newton was not anonymous, for Leibniz had mentioned him in his introductory note. Leibniz was so embarrassed by the whole thing that he wrote the Royal Society indicating that he was not the author of the challenge problems. Technically, this was true, but he had contrived with Bernoulli to embarrass Newton.

Within a few years there were solutions by John Craige, David Gregory, Richard Sault and Fatio de Duillier. In 1704 Charles Hayes, in his widely read *Treatise on Fluxions*, presented it as a mere worked example in this textbook. As often happens, a difficult problem, once cleverly solved, comes within the grasp of many.

Perhaps the most important of all of these solutions is that by Jakob Bernoulli. While somewhat ponderous, it led to a new field of mathematics, the calculus of variations, a branch of analysis where the variables are not numbers but functions.

The clever elementary solution of Johann Bernoulli is the one that I like to present at the end of a course on the integral calculus, for it provides both a wonderful story and a great review. You can find it in many places, e.g., [10, 12]. Translations of the solutions of the Bernoulli brothers can be found in [13].

Building your brachistochrone

One of my students, Zachary Seidel, had a brachistochrone shaped sliding board as a young child. His father had studied mathematics as an undergraduate and decided that his son should have the quickest slide in the neighborhood. Such a slide is probably too big for your classroom, so the model designed by Cliff Long will be described.

You will need a sheet of 3/4 inch plywood that measures roughly 30 by 14 in. First cut a strip off the long edge about 1 1/4 inches wide; its use will be described momentarily. From another piece of plywood, cut out a circle of radius 4.5 in. Drill a hole in the center and insert a 1/4 inch dowel that sticks up far enough to provide a nice handle. Near the circumfrence of the disk drill another hole that will hold a small piece of chalk (ideally the chalk would be *at* the circumference, but that makes the construction of our bachisto-chrone harder). The distance from the center of this hole to the circumference of the disk should be slightly more than the radius of the chalk. This is so the chalk will not fall out. You should also drill another hole in the disk at about half the radius so the chalk can be moved there later to draw a curtate cycloid.

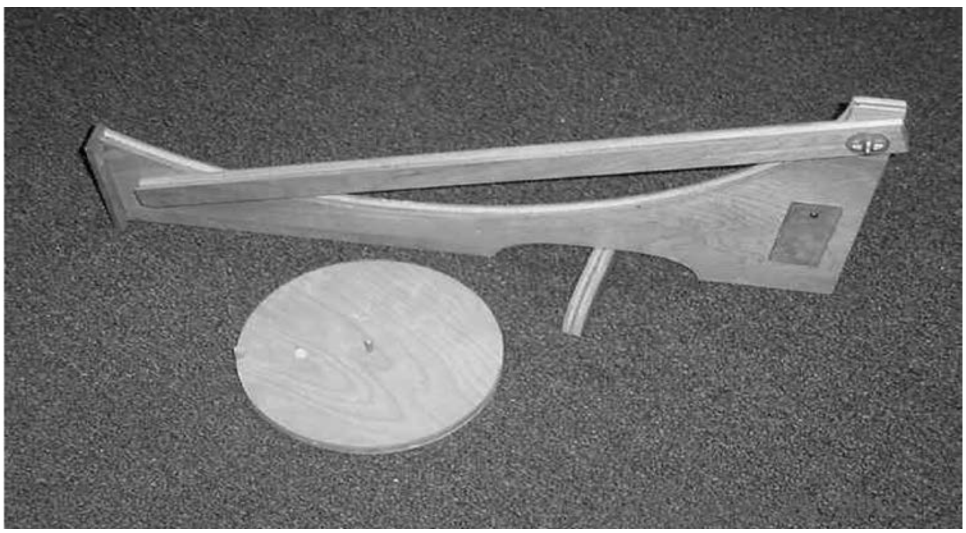

Figure 4. The Finished Brachistochrone

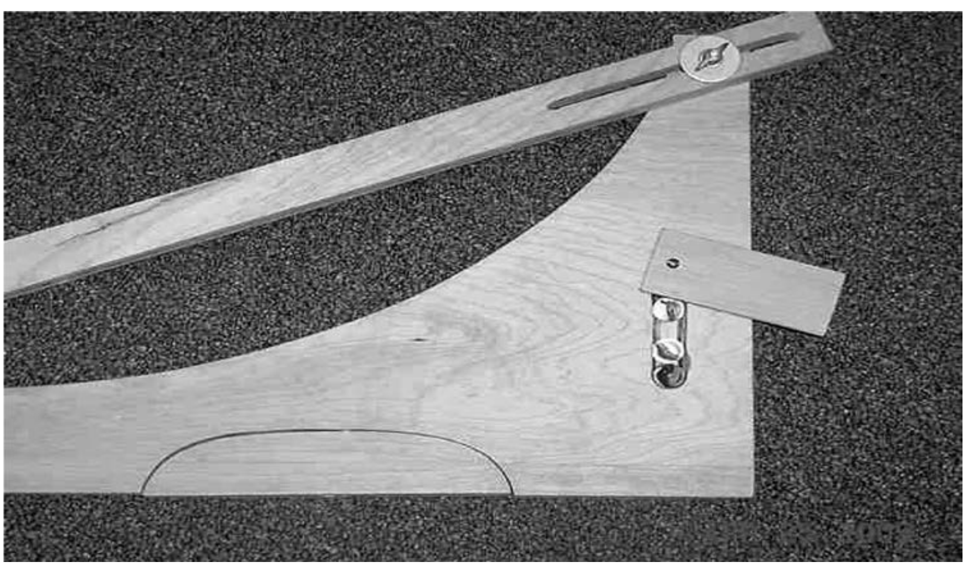

Figure 5. Details of construction.

Use this disk and chalk to trace out one arch of the cycloid on the sheet of plywood. Start with the chalk about 2 inches from the short edge and on the same side where you cut out the disk. Now cut on this chalk line and you will have a cycloid. Use your router to cut out a groove on the edge of the plywood so you will have a track for a marble to run down.

The fun part of this model is a straight line track for another marble to run down. It will be used to demonstrate clearly that the straight line is not the quickest path. To do this, you will use the piece of plywood that you cut off earlier. Cut a groove on one side for the marble. Cut a slot along the middle of this board starting about one inch from the end and about 12 inches long. The slot should be wide enough so that it will slide over a set screw. The idea is that you want to adjust where the straight line track ends. At one extreme it should be possible to go almost horizontally across the whole arch of the cycloid; on the other, you want to be able to adjust it so that the straight line track ends at the low point on the inverted cycloid, and perhaps even before that.

Next cut out a semi-elliptical piece on the yet untouched straight base. With a screw, and washer used as a spacer, this will provide a toggle stand for your brachistochrone. Finally, you need a place to store several marbles (three are suggested in case you lose some of your marbles in class). See Figure 5.

Using your brachistochrone in class

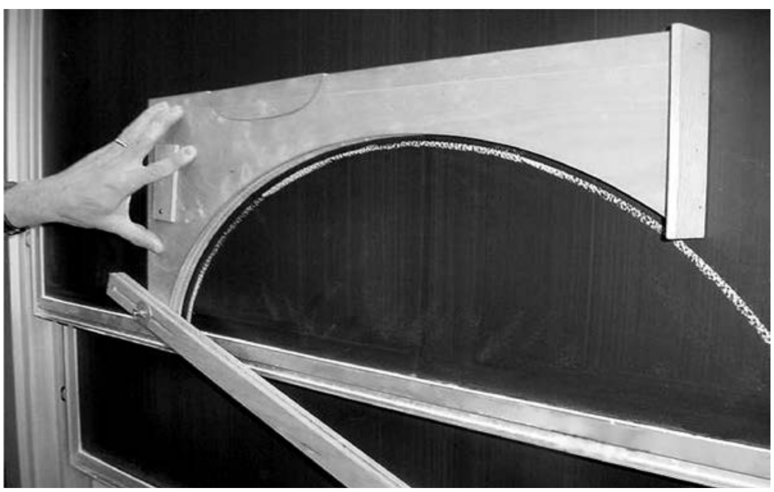

Figure 6. Brachistochrone = Cycloid.

After you have drawn a cycloid on the chalkboard, and before you have derived its equation, hold your brachistochrone up to the chalkboard to show the students that the curve you just drew is the same curve as your brachistochrone. Now you can explain Johann Bernoulli's problem and let them test the cycloid solution against the straight line solution.

This is an opportunity to get your students involved. Invite several of them to adjust the straight line chute so that it ends at the low point of the inverted cycloid on the brachistochrone. Have one of them get out two marbles and hold them with one hand so that one marble is on the brachistochrone track and the other is on the straight line track. By using one hand it is easier to release the marbles simultaneously. Some chaos will result when the marbles go flying off the end of the tracks and across the room. Probably some students will be disbelievers—when one first encounters this demonstration it is hard to believe that the cycloid track is quicker than the straight line track—and want to try themselves. This is good; get everyone involved.

Next adjust the straight chute to direct one marble to another point. Pick a point that is beyond the low point of the brachistochrone, on the uphill slope. This time, to the astonishment of the students, the marble race is even more unfair. Travel along the brachistochrone is much quicker. Students are perplexed that the quickest way to get from A to B is to dip below B and then coast back up to it. They will ask why this happens. They do not realize that they are asking to see the proof that the brachistochrone really is the quickest path. This shows the motivational power of interesting historical examples combined with a tactile demonstration of the result.

I have met mathematicians who believed that to get from A to B on the quickest path, that one should draw a cycloid that begins at A and has its low point at B. To see that this is incorrect, examine Newton's picture again (Figure 3). A little thought will show that it is rarely the case that B is at the bottom of the cycloid. If the points are far apart, it is impossible to get from the upper to the lower point without dipping below the lower point and then coasting back up.

The brachistochrone model can be used to demonstrate another interesting property of the cycloid. The curve is isochronous, i.e., no matter where one starts on the curve the time it takes to reach the low point is the same (see [2] for the original proof). To illustrate this, have a student hold a marble in each hand, one on each side of the minimum point on the cycloid, but at different distances from the midpoint, and then release them simultaneously. They will collide at the low point of the cycloid. To confirm this the observer needs to be directly in front of the model and to focus on the minimum point of the cycloid (a mark on the model will help locate this point for the observer). Your students will want to repeat this multiple times from different starting positions thereby giving all a chance to observe up close. Andy Long, Cliff's son, suggests another way to do this. Ask your students to close their eyes and remain quiet so that they can hear the equal beats of a single marble rolling back and forth on the cycloidal track. Use a large amplitude so that it goes back and forth a number of times. Your students will unconsciously rock their heads back and forth in time with this cycloidal clock.

The ordinary pendulum is not isochronous; the period T depends upon both the length L and the angle of oscillation θ. This provides an interesting real world example of a multivariable function, which is due to Daniel Bernoulli in 1749.

$$T(L,\theta)=2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1^2}{2^2}\sin^2\frac{\theta}{2}+\frac{1^2+3^2}{2^2+4^2}\sin^4\frac{\theta}{2}+\cdots\right),$$

When the angle of oscillation is small, then all of the terms of the series involving the sine can be ignored. This provides yet another point in the curriculum to bring in this circle of ideas.

Christiaan Huygens, the foremost mathematical physicist in the generation before Newton, took advantage of this property of the cycloid to design an accurate pendulum clock (Figure 7). But he needed one more mathematical property of the cycloid: the evolute of a cycloid is another cycloid of the same size.

He hung the bob of his pendulum on a thread that swung between cycloidal cheeks. The cheeks prevented the bob from swinging in a circular arc like in a regular pendulum. When the bob moved to the side it wound around the cycloidal cheeks and was pulled up slightly. He showed that the curve of the bob was a cycloid.

Huygens published this work in his *Horologium oscillatorium* of 1673; you can read how he constructed a cycloid (he had a clever device for avoiding slippage) and designed his clock [2]. A model of his clock can be seen in the Borehaave, a wonderful science museum in Leiden in the Netherlands. Unfortunately, as one might suspect,

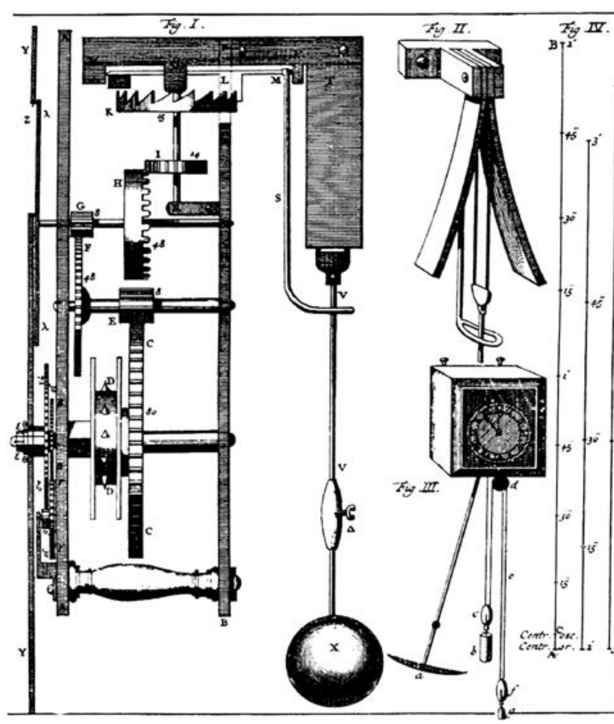

Figure 7. The Pendulum Clock of Christiaan Huygens

the accuracy of Huygens' clock was killed by friction. This illustrates the old adage: In theory, theory and practice are the same, but in practice, theory and practice are not the same.

Trains and epicycloids

A rewarding question to pose to students is this: As you watch a speeding train go by, what point on it is moving backwards? It will probably take some prompting to get them to realize that the answer is the bottom of each wheel, the flange, which is the part of the wheel below the top of the rail and which keeps the train on the track. This will give us another opportunity to use our device, but we will need another piece to attach to our cycloid drawer, a piece of wood several inches longer than the radius of our disk. It should have a hole which goes over the peg at the center of the disk, another peg to go in the hole where the chalk went while drawing the cycloid, and another hole near the end for another piece of chalk. This arrangement forces the board to turn as the disk turns. If the chalk is at distance $d > r$ from the center of the circle, then it produces a prolate cycloid. When you use this device to draw a prolate cycloid in class, your students will literally be able to see that when the chalk is at the bottom of the curve it is moving backwards.

The student who understands how the parametric equations for the ordinary cycloid were obtained will have no trouble finding the equations for a prolate cycloid:

$$\begin{cases} x = r\theta - d\sin\theta, \\ y = r - d\cos\theta. \end{cases}$$

If we compute the derivative in the x-direction we obtain

$$\frac{dx}{d\theta} = y = r - d\cos\theta$$

If θ is (near) an even multiple of π, then this derivative is negative. In practical terms, the bottom of the train wheel is moving backwards.

When d is arbitrary the curve is called a trochoid, when $d > r$ it is a prolate cycloid and when $d < r$ it is a curtate cycloid. Curtate cycloids are used by some violin makers for the back arches of some instruments, and they resemble those found in some of the great Cremonese instruments of the early 18th century, such as those by Stradivari [8].

Conclusion

One does not have to prove everything one talks about. We mathematicians need to start talking about mathematics. There are so many interesting things in mathematics that will captivate students. Once learners are interested they will be motivated to get a deeper understanding of that mathematics. They will seek out the proofs on their own.

We have seen the essential ingredients of good teaching combined in one problem. It is a problem with a fascinating history, connected to a multitude of important names that our students should know. The mathematics is tractable to undergraduates. The ideas can be used at several places in the curriculum and this spiral approach enhances learning. Finally one has a nice classroom device. For all of this, I thank Cliff Long.

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References

1. Thomas Hern, Cliff Long and Andy Long, "Looking at Order of Integration and a Minimal Surface," *The College Mathematics Journal*, 29 (1998) 128–133.
2. Christiaan Huygens, *Horologium oscillatorium* (1673). English translation as *Christiaan Huygens' the pendulum clock, or, Geometrical demonstrations concerning the motion of pendula as applied to clocks*, translated with notes by Richard J. Blackwell; introduction by H. J. M. Bos, Ames: Iowa State University Press, 1986.
3. Cliff Long, "Peaks, Ridge, Passes, Valley and Pits: A Slide Study of $f(x, y) = Ax^2 + By^2$," *The American Mathematical Monthly*, 83 (1976) 370–371.
4. ———, "A Flexible Model for Peak, Ridge and Pass," *The College Mathematics Journal*, 7 (1976) 16–17.
5. ———, 1931–2002, http://www.bgsu.edu/departments/math/Ohio-section/CliffLong/
6. ———, *Bug on a [Möbius] Band*: http://www.wcnet.org/~clong/carving/bug.GIF
7. ———, The Frenet frame. http://www.wcnet.org/~clong/carving/3knotsandtwist.GIF
8. (Playfair 1990). http://mathworld.wolfram.com/CurtateCycloid.html
9. Rickey, V. Frederick, "Mathematics of the Gregorian calendar," *The Mathematical Intelligencer*, 7(1985), no. 1, 53–56.
10. ———, "History the Brachistochrone," to appear in *Historical Notes for Calculus Teachers*. Available at http://www.dean.usma.edu/departments/math/people/rickey/hm/CalcNotes/brachistochrone.pdf
11. J. F. Scott, ed. *The Correspondence of Isaac Newton*, Vol. IV, 1694–1709, Cambridge University Press, 1967. See letters #561 to Montague on 30 Jan 1969/7 (pp. 220–229) and #601 to Flamsteed 9 Jan 1698/9 (296–7), which are in English and Latin.
12. George F. Simmons, *Differential Equations with Applications and Historical Notes*, McGraw-Hill, 1972. See pp. 25–31.
13. Dirk J. Struik, *A Source Book in Mathematics, 1200–1800*, Harvard University Press, 1969. Contains English translations of the solutions of the brachistochrone problem by Johann and Jakob Bernoulli, pp. 391–399.
14. Rüdiger Thiele, "The brachistochrone problem and its sequels," unpublished manuscript of a lecture given at the Frederik Pohle Colloquium, Adelphi University, January 24, 2001.
15. Richard S. Westfall, *Never at Rest. A Biography of Isaac Newton*, Cambridge University Press, 1980. See pp. 581–583.
16. D. T. Whiteside, *The Mathematical Papers of Isaac Newton, Volume VIII, 1697–1722*, Cambridge University Press, 1981. Contains excellent and very scholarly commentary by Whiteside (pp. 1–14) as well as the pertinent documents (pp. 72–90) and numerous references.
17. E. A. Whitman, "Some historical notes on the cycloid," *The American Mathematical Monthly*, 50 (1943), 309–315. Reprinted in *Sherlock Holmes in Babylon and Other Tales of Mathematical History*, MAA, 2004, pp. 183–187.