Taylor Polynomials

MAT 229, Spring 2021

Week 13

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 11.11: Applications of Taylor polynomials
- Boelkins/Austin/Schlicker's Active Calculus
  Section 8.5: Taylor polynomials and Taylor series

Taylor series review

If function $f(x)$ has a power series representation centered at a, then that power series must be the Taylor series centered at a,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Question

Let $f(x) = \sin(x)$.

- What is the Taylor series for $f(x)$ centered at π?
- What is its interval of convergence?

(Video)

- We have already seen that the Taylor series for $f(x)$ centered at 0 is $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$. Replace x in this with $x - \pi$. How do the results of this series compare with the Taylor series for $f(x)$ centered at π?
- There is a trigonometric identity that $\sin(x - \pi) = -\sin(x)$. What does this mean for the above calculations?

(Video)

Tangent line approximation

Question

Given a general function $f(x)$ and value $x = a$, what is the tangent line approximation of $f(x)$ at a?

(Video)

Questions

Let $f(x) = \cos(x) + \sin(2x)$.

- What is an equation of the tangent line to $y = f(x)$ at $x = 0$?
- What is the sum of the linear terms of the series for cos(x) and sin(2x) at 0?

(Video)

Taylor polynomials

Definition

The **n^{th} degree Taylor polynomial of $f(x)$ centered at a** is the partial sum of the Taylor series that goes up to and includes the n^{th} power of $(x - a)$. If the Taylor series for $f(x)$ centered at a converges to $f(x)$ for a given value x, then the n^{th} Taylor polynomial of $f(x)$ centered at a provides a polynomial approximation to $f(x)$.

Questions

Let $f(x) = \cos(x) + \sin(2x)$.

- What is the first degree Taylor polynomial of $f(x)$ centered at 0?
- What is the second degree Taylor polynomial of $f(x)$ centered at 0?
- What is the third degree Taylor polynomial of $f(x)$ centered at 0?

(Video)

- Plot the graphs of three polynomials along with the graph $y = f(x)$.
- Using the third degree Taylor polynomial of $f(x)$ centered at 0, approximate $f(0.5)$.

(Video)

Taylor remainder

If you approximate a quantity, you need some way to analyze how good the approximation is. Consider the error = | approximation − exact | . Here the exact is a given function $g(x)$ and the approximation is the n^{th} degree Taylor polynomial of $g(x)$ centered at a.

Definition

The **n^{th} Taylor remainder of $g(x)$ centered at a** is

$$R_n(x) = g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k$$

In other words the absolute error in using the n^{th} degree Taylor polynomial to approximate the function is $|R_n(x)|$.

Analyzing error

The Taylor series error estimate: If $\left| f^{(n+1)}(x) \right| \leq M$ for all values of x of interest, then

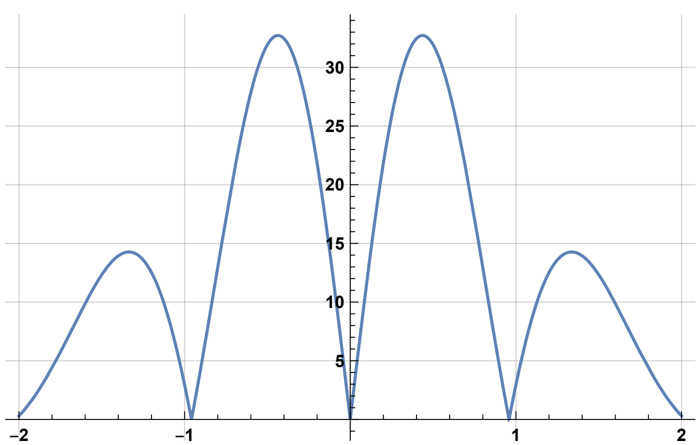
$$\left| R_n(x) \right| \leq \frac{M}{(n+1)!} \left| x-a \right|^{n+1}.$$

You can think of this roughly this way: the error on the interval is smaller than the largest “first neglected Taylor term” evaluated anywhere on the interval.

Questions

Let $f(x) = e^{-x^2}$.

- What is the Taylor series centered at zero for e^x?
- Using the Taylor series for e^x what is the Taylor series for $f(x)$?
- What is the fourth degree Taylor polynomial for $f(x)$ centered at 0?
- From the plot of $\left| f^{(5)}(x) \right|$ shown below, use the Taylor series error estimate to bound the error in approximating $f(x)$ with the 4^{th} degree Taylor polynomial on the interval [-2,2].



(Video)