Infinite Sequences

MAT 229, Fall 2021

Week 9

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  11.1: Sequences
- Boelkins/Austin/Schlicker's Active Calculus
  8.1: Sequences

Review

Questions

- Evaluate $\int_1^\infty \frac{1}{x^\pi}\,dx$ and $\int_1^\infty \frac{1}{x^{0.1}}\,dx$.
- For which exponents p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?
- Does $\int_1^\infty \frac{1}{x+x^2}\,dx$ converge?

Questions

- Evaluate $\int_0^1 \frac{1}{x^\pi}\,dx$ and $\int_0^1 \frac{1}{x^{0.1}}\,dx$.
- For which exponents p does $\int_0^1 \frac{1}{x^p}\,dx$ converge?
- Does $\int_0^1 \frac{1}{x^2-x\sin(x)}\,dx$ converge?

Approximations

Questions

Use the trapezoid rule to approximate $\int_0^{\pi^2} \cos\left(\sqrt{x}\right)\,dx$

- using $n = 1$
- using $n = 2$
- using $n = 3$
- using $n = 4$
- What do we expect to happen as we use larger and larger values of n in the trapezoid rule?

Questions

A standard way to approximate the square root of a value is as follows. This method was known to the Babylonians, a few thousand years ago.... To approximate $\sqrt{a}$:

1. Make a rough estimate for the value of $\sqrt{a}$. Call it the first approximation x_1.
2. The second approximation is $x_2 = 0.5\left(x_1 + \frac{a}{x_1}\right)$.
3. The third approximation is $x_3 = 0.5\left(x_2 + \frac{a}{x_2}\right)$.
4. In general, the n^{th} approximation is $x_n = 0.5\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$.

- Use this scheme to find the fourth approximation to $\sqrt{2}$ using $x_1 = 2$.
- Do the approximations improve as n gets larger?

This method is an example of "Newton's method" (section3.8), which is a very general method for finding roots (zeros) of functions.

Sequences

Definition

A *sequence* is an infinite list of terms (frequently numbers, but we'll generalize that!) written in a definite order. We usually start counting from 1:

$$a_1, a_2, a_3, \ldots, a_k, \ldots$$

Examples

- $1, 2, 3, 4, 5, \ldots$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
- $-1, 2, -3, 4, -5, 6, -7, 8, \ldots$

As an example of a generalization of number sequences, consider Taylor polynomials:

$$T_n(x) = \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!} (x-a)^k$$

We can talk about the **sequence of Taylor polynomials** for *g(x)=sin(x)* about center a=0: $\{T_1(x), T_2(x), T_3(x), \ldots.\}$. So we can have sequences of functions, too! And we'll be generalizing other notions of numbers sequences down the road.

Forms

Besides writing a sequence as a list we will also write some sequences

- as formulas: for example $\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$=$\{\frac{1}{2^1},\frac{2}{2^2},\frac{3}{2^3},....\}$
- recursively: for example $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$: {1,1,2,3,5,8,13,21,34,....}

Limits

Frequently, but not always, sequences $\{a_n\}$ are lists of approximations that converge to some desired exact values. If the limit of a_n as n tends to infinity exists,

$\lim_{n\to\infty} a_n$ exists

then we say that the sequence *converges*; otherwise it *diverges*.

Questions

Do any of the above sequences converge? If so, to what values?

- $x_1 = 2$, $x_n = 0.5\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$
- $-1, 2, -3, 4, -5, 6, -7, 8, \ldots$
- $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \ldots$
- $\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$
- $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$

Techniques (for deciding convergence)

Technique 1

If $a_n = f(n)$ where f is a real-valued function, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Questions

- Find $\lim_{n\to\infty} \frac{n}{n^2+1}$ by evaluation the limit $\lim_{x\to\infty} \frac{x}{x^2+1}$.
- For which of the above sequences can you use this technique?

Technique 2

If $\lim_{n\to\infty} | a_n | = 0$, then $\lim_{n\to\infty} a_n = 0$.

Question

- Does this technique apply to any of the above limits?
- For which values of r does the sequence $\{r^n\}$ converge?

Monotonic sequences

Definition

An *increasing* sequence is one whose terms a_n satisfy $a_{n+1} \geq a_n$ for all n. It is *strictly increasing* if we can write $a_{n+1} > a_n$ for all n.

Similarly, a *decreasing* sequence is one whose terms a_n satisfy $a_{n+1} \leq a_n$ for all n. It is *strictly decreasing* if we can write $a_{n+1} < a_n$ for all n.

- What is an example of an increasing sequence?
- What is an example of an decreasing sequence?
- What is an example of a sequence that is neither increasing nor decreasing?

Definition

A sequence that is either increasing or decreasing is said to be *monotonic*.

Bounded sequences

We're familiar with bounds from our discussion of errors in numerical integration. There we established a bound on an error -- a value which was certainly larger than the error made in approximating the integral (I think of it as a "tolerance" -- how much slop you're allowed in an answer!).

- What should we mean by a sequence that is *bounded above*?

(Hopefully you said that there's a value greater than every entry in the sequence.)

- What is an example of a sequence that is bounded above?
- What should we mean by a sequence that is *bounded below*?
- What is an example of a sequence that is bounded below?

Definition

A *bounded* sequence that is bounded above **and** below.

Technique 3

A bounded, monotonic sequence converges.

You might think this way: *the sequence has to go somewhere -- it's constantly increasing (say), and yet it can't go past a certain point -- so it has to go somewhere between where it is at any moment and that* *upper bound.* Furthermore, this is true even as long as the sequence is **eventually** monotonic.

You might think this way: *no finite chunk of a sequence at the beginning has any impact on convergence. Convergence is a property of the long tail, as the index sails off to infinity.*

The same is true for improper integrals, whose domain includes -∞ or ∞: it's all about the tail. You can rewrite one of these integrals, e.g. $\int_1^\infty \frac{1}{x^p}\,dx$, as $\int_1^b \frac{1}{x^p}\,dx + \int_b^\infty \frac{1}{x^p}\,dx$; the first integral is proper -- no question about its convergence -- while the second is still improper. It's the "improper part", no matter what value of $b>1$.

Questions

Consider the sequence $\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$.

- Is this sequence monotonic?
- Is this sequence bounded?

Homework

- Weekly assignment 7
- IMath problems on sequences