

# Power Series

MAT 229, Spring 2021

Week 12

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## Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*  
Section 11.8: Power Series
- Boelkins/Austin/Schlicker's Active Calculus  
Section 8.6: Power Series

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## Review

Two tests that will be the workhorses for our analysis are based on comparisons to geometric series.

### Ratio test

Given any series  $\sum_k b_k$ , evaluate the limit of the ratio of consecutive terms, ignoring any signs,

$$\lim_{k \rightarrow \infty} \left| \frac{b_{k+1}}{b_k} \right|.$$

- If  $\lim_{k \rightarrow \infty} \left| \frac{b_{k+1}}{b_k} \right| = L$  where  $L < 1$ , then  $\sum_k b_k$  converges absolutely.
- If  $\lim_{k \rightarrow \infty} \left| \frac{b_{k+1}}{b_k} \right| = L$  where  $L > 1$ , then  $\sum_k b_k$  diverges.
- If  $\lim_{k \rightarrow \infty} \left| \frac{b_{k+1}}{b_k} \right| = L$  where  $L = 1$ , then you must use another convergence test.

### Root test

Given any series  $\sum_k b_k$ , evaluate the limit of the  $k^{\text{th}}$  root, ignoring any signs, of the  $k^{\text{th}}$  term,

$$\lim_{k \rightarrow \infty} \sqrt[k]{|b_k|}.$$

- If  $\lim_{k \rightarrow \infty} \sqrt[k]{|b_k|} = L$  where  $L < 1$ , then  $\sum_k b_k$  converges absolutely.
- If  $\lim_{k \rightarrow \infty} \sqrt[k]{|b_k|} = L$  where  $L > 1$ , then  $\sum_k b_k$  diverges.

- If  $\lim_{k \rightarrow \infty} \sqrt[k]{|b_k|} = L$  where  $L = 1$ , then you must use another convergence test.

## Power series

How can we approximate  $\sin(1)$  or  $e^{-1.5}$ ? We can represent both of these as **power series**, which are essentially series with polynomials as terms rather than just constants. So they're functions, written as infinite sums of "monomials". I hope that you're thinking "Cool!" -- but you might be thinking "Oh my God!" ...)

Turns out that a lot of our familiar functions can be written this way; furthermore, we can then approximate particular values of the functions using partial sums. In particular, many transcendental functions, like  $\sin(x)$  and  $e^x$ , can be written as power series in  $x$ : **polynomials of infinite degree**.

(As you know, nothing excites me like infinity! I get infinitely excited....)

### Definition

A *power series* in  $x$  centered at  $a$  is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

where  $c_n$  is an expression in  $n$  and  $a$  is a number.

### Questions

Identify the  $c_n$  and  $a$  for each of the following power series.

- $\sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n}$
- $\sum_{n=1}^{\infty} \frac{(2n-1)}{n^2} x^n$
- $\sum_{n=0}^{\infty} (3x)^n$
- $\sum_{n=0}^{\infty} \frac{(5x-3)^n}{n!}$

([Video](#))

### Questions

A function is defined as the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = 1 + \frac{x-1}{2} + \frac{(x-1)^2}{4} + \frac{(x-1)^3}{8} + \dots$$

- What is  $f(1)$ ?
- What is  $f(2)$ ?
- Is  $f(3)$  defined? If so, what is it? If not, why not?

- To determine the domain of this function, what do we need to worry about?

(Video)

## Terminology

- The *interval of convergence* is the domain for a power series.
- The *radius of convergence* is the distance from the center to the edge of the interval of convergence. It comes from the ratio/root test.

## Questions

A function is defined as the power series

$$g(x) = \sum_{n=1}^{\infty} \frac{3(x+4)^n}{n}$$

- What is the center of this series?
- For what value of  $x$  is it easy to evaluate this function?
- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?

(Video)

## Questions

$$\text{Let } H(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- What are the first 4 terms of the series?
- What is the center of this series?
- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?

(Video)

- Using an appropriate error estimate approximate  $H(-1)$  with error less than 0.0001. (Video)
- What is this function? Let's plot some partial sums....
- Can you guess the function?

(Video)

## Questions

$$\text{Let } F(x) = \sum_{n=0}^{\infty} \frac{(2x+3)^n}{n^2+1}.$$

- What are the first 4 terms of the series?
- What is the center of this series?

- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?

[\(Video\)](#)

- Using an appropriate error estimate approximate  $F(-1)$  with error less than 0.0001. [\(Video\)](#)
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## Homework

- IMath problems on power series.