Trigonometric Integration

MAT 229, Spring 2021

Week 4

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 7.2: Trigonometric integrals
- Boelkins/Austin/Schlicker's Active Calculus
  Section 5.3: Integration by substitution

Review

We have three antidifferentiation techniques:

1. Recognize it immediately as the derivative of some function, for example
   $\int e^x \, dx = e^x + C$
2. Try substituting for some internal part (the chain rule backwards). Don't forget the differential. For example, to evaluate $\int x^2 \cos(x^3) \, dx$ use the substitution $u = x^3$ and $du = 3 x^2 \, dx$ to replace the above integral with the simpler one,
   $\int \frac{1}{3} \cos(u) \, du = \frac{1}{3} \sin(u) + C = \frac{1}{3} \sin(x^3) + C$
3. Integration by parts (the product rule backwards). Here it is in two different forms:
   a. $\int u \, dv = u \, v - \int v \, du$
   b. $\int u(x) \, v'(x) \, dx = u(x) \, v(x) - \int v(x) \, u'(x) \, dx$

Today we introduce trigonometric integrals, as a special class of integrals that seem to arise frequently, and that involve particular trig substitutions.

Questions

Choose an appropriate technique to evaluate the following integrals.

- Find the area of the region bounded above by $y = x \, e^{-x}$, below by the x-axis, and the line $x = 5$.
- Find the volume of the surface of revolution generated by rotating about the x-axis the region bounded above by $y = x \sec(x^3)$, below by the x-axis, and the line $x = 1$.

Trigonometric identities

Pythagorean identities

$\cos^2(\theta) + \sin^2(\theta) = 1$

Questions

- Divide this equation by $\cos^2(\theta)$ to get another Pythagorean identity.
  $\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$
  $\longrightarrow 1 + \tan^2(\theta) = \sec^2(\theta)$
- Divide this equation by $\sin^2(\theta)$ to get another Pythagorean identity.
  $\frac{\cos^2(\theta)}{\sin^2(\theta)} + \frac{\sin^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$
  $\longrightarrow \cot^2(\theta) + 1 = \csc^2(\theta)$

Double angle formulas

There are two identities (other than the Pythagorean) that I think are worth memorizing: the sine and cosine **sum** formulas are

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

You can get the difference formulas from the sum formulas, using the symmetry properties of sines and cosines (so you don’t need to memorize those -- but you do need to know those symmetry properties!):

$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$.

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Questions

- What is $\sin(2\theta) = \sin(\theta + \theta)$? (Video)
- What is $\cos(2\theta) = \cos(\theta + \theta)$? (Video)
- Rewrite this using a Pythagorean trigonometric identity so that $\cos(2\theta)$ is in terms of only $\cos(\theta)$.
- Rewrite this using a Pythagorean trigonometric identity so that $\cos(2\theta)$ is in terms of only $\sin(\theta)$. (Video)

Sine squared and cosine squared

The results above give us two trigonometric identities that we will use below.

- $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

Trigonometric integrals

Here are some techniques for evaluating integrals of the following forms.

- $\int \sin^m(x)\cos^n(x)\,dx$
- $\int \tan^m(x)\sec^n(x)\,dx$
- $\int \cot^m(x)\csc^n(x)\,dx$

Questions/review:

Integration is just differentiation in reverse: what are the following derivatives?

- $\frac{d}{dx}\cos(x)$
- $\frac{d}{dx}\sin(x)$
- $\frac{d}{dx}\tan(x)$
- $\frac{d}{dx}\sec(x)$
- $\frac{d}{dx}\cot(x)$
- $\frac{d}{dx}\csc(x)$

Questions

- Evaluate $\int \cos^4(x)\sin(x)\,dx$ using the substitution $u=\cos(x)$.(Video)
- Evaluate $\int \sin^6(x)\cos^3(x)\,dx$ by writing this as $\int \sin^6(x)\cos^2(x)\cos(x)\,dx = \int \sin^6(x)\left(1-\sin^2(x)\right)\cos(x)\,dx$. (Video)
- Find the area under the curve $y=\sin^3(x)\cos^3(x)$ for $0 \le x \le \pi/2$. (Video)
- Evaluate $\int \cot^4(x)\csc^2(x)\,dx$. (Video)
- Evaluate $\int \tan^2(x)\sec^4(x)\,dx$ by first writing $\sec^4(x)=\sec^2(x)\sec^2(x)$, using a Pythagorean identity on one of the factors, and then making an appropriate substitution. (Video)

Questions

- Use the trigonometric identity $\cos^2(\theta)=\frac{1+\cos(2\theta)}{2}$ to evaluate $\int \cos^2(\theta)\,d\theta = \int \frac{1+\cos(2\theta)}{2}\,d\theta$. (Video)
- Find the volume of the solid of revolution obtained by rotating about the x-axis the region above the x-axis and one arch of the sine curve $y=\sin(x)$. (Video)

Homework

IMath problems on trigonometric integrals.