Trigonometric Integrals and Substitution

MAT 229, Spring 2021

Week 5

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 7.3: Trigonometric Substitution
- Boelkins/Austin/Schlicker's Active Calculus
  Section 5.3: Integration by substitution

Review

We use some trig identities to conquer these formerly difficult integrals (e.g. Pythagorean):

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

There are two other identities that I think are worth memorizing: the sine and cosine **sum** formulas.

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Others can be easily derived from these, e.g. double angle:

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

and identities for tangent, secant, etc.

Products of sine and cosine:

To integrate $\int \sin^n(x)\cos^m(x)\,dx$

- If n is a positive odd integer, write this integral as
  $$\int \sin^n(x)\cos^m(x)\,dx = \int \sin^{n-1}(x)\cos^m(x)\sin(x)\,dx$$
  and use the substitution $u = \cos(x)$ so that $du = -\sin(x)\,dx$. Since n is odd, $n - 1$ is even so that

$n-1=2k$.

$$\sin^{n-1}(x)=\sin^{2k}(x)=(\sin^2(x))^k=(1-\cos^2(x))^k.$$

- If m is a positive odd integer, do the same thing as the above using the substitution $u=\sin(x)$.
- If n and m are positive even numbers use the trigonometric identities

$$\sin^2(\theta)=\frac{1-\cos(2\theta)}{2}$$

$$\cos^2(\theta)=\frac{1+\cos(2\theta)}{2}$$

to reduce the power on the trig functions.

Questions

- What is the value of $\int \sin^2(3x)\,dx$?
- What is the value of $\int \sin^2(x)\cos^2(x)\,dx$?

Products of secant and tangent:

To integrate $\int \sec^n(x)\tan^m(x)\,dx$

- If n is a positive even integer, write this integral as

$$\int \sec^n(x)\tan^m(x)\,dx=\int \sec^{n-2}(x)\tan^m(x)\sec^2(x)\,dx$$

and use the substitution $u=\tan(x)$ so that $du=\sec^2(x)\,dx$. Since n is even, $n-2$ is even so that $n-2=2k$.

$$\sec^{n-2}(x)=\sec^{2k}(x)=(\sec^2(x))^k=(1+\tan^2(x))^k.$$

- If m is a positive odd integer, write this integral as

$$\int \sec^n(x)\tan^m(x)\,dx=\int \sec^{n-1}(x)\tan^{m-1}(x)\tan(x)\sec(x)\,dx$$

and use the substitution $u=\sec(x)$ so that $du=\sec(x)\tan(x)\,dx$. Also, use the trigonometric identity $\tan^2(x)=\sec^2(x)-1$.

Questions

Evaluate $\int \tan^5(x)\sec^4(x)\,dx$.

Simple trigonometric integrals

You know the values of $\int \cos(x)\,dx$ and $\int \sin(x)\,dx$.

- Write $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$ and use a substitution to evaluate $\int \tan(x)\,dx$.
- What is $\int \cot(x)\,dx$?
- Write $\sec(x)$ as $\sec(x)\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$ and use a substitution to evaluate $\int \sec(x)\,dx$.
- What is $\int \csc(x)\,dx$?

Trigonometric identities

Pythagorean identities

$\cos^2(\theta) + \sin^2(\theta) = 1 \quad \longrightarrow \quad \cos^2(\theta) = 1 - \sin^2(\theta)$

This means for any number $a > 0$,

$$\sqrt{a^2 - a^2 \sin^2(\theta)} = \sqrt{a^2 \cos^2(\theta)} = \pm a \cos(\theta)$$

Questions

- Simplify $\sqrt{a^2 + a^2 \tan^2(\theta)}$.
- Simplify $\sqrt{a^2 \sec^2(\theta) - a^2}$.

Trigonometric substitutions

- If an integral involves $\sqrt{a^2 - x^2}$, use the substitution $x = a \sin(\theta)$.
- If an integral involves $\sqrt{a^2 + x^2}$, use the substitution $x = a \tan(\theta)$.
- If an integral involves $\sqrt{x^2 - a^2}$, use the substitution $x = a \sec(\theta)$.

Example

To evaluate $\int x^3 \sqrt{4 - x^2}\, dx$ use the substitution $x = 2 \sin(\theta)$ with $-\pi/2 \leq \theta \leq \pi/2$. That means $dx = 2 \cos(\theta)\, d\theta$.

$$\int x^3 \sqrt{4 - x^2}\, dx = \int 8 \sin^3(\theta) \sqrt{4 - 4 \sin^2(\theta)}\; 2 \cos(\theta)\, d\theta$$

$$= \int 8 \sin^3(\theta) \sqrt{4 \cos^2(\theta)}\; 2 \cos(\theta)\, d\theta$$

$$= \int 8 \sin^3(\theta)\, 2 \cos(\theta)\, 2 \cos(\theta)\, d\theta$$

$$= 32 \int \sin^3(\theta) \cos^2(\theta)\, d\theta$$

$$= 32 \int \sin^2(\theta) \cos^2(\theta) \sin(\theta)\, d\theta$$

$$= 32 \int (1 - \cos^2(\theta)) \cos^2(\theta) \sin(\theta)\, d\theta$$

Now make the substitution $u = \cos(\theta)$ so that $du = -\sin(\theta)\, d\theta$ or $\sin(\theta)\, d\theta = -du$.

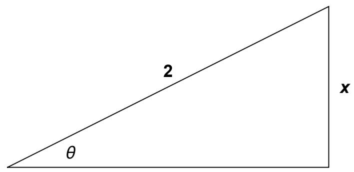
$$= -32 \int (1 - u^2)\, u^2\, du$$

$$= -32 \int (u^2 - u^4)\, du$$

$$= -32 \left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= -32 \left(\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5}\right) + C$$

Since $x = 2 \sin(\theta)$, $\sin(\theta) = x/2$



From the corresponding right triangle, $\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$. The final answer is

$$-32\left(\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5}\right) + C = -32\left(\frac{\left(\sqrt{4-x^2}/2\right)^3}{3} - \frac{\left(\sqrt{4-x^2}/2\right)^5}{5}\right) + C$$

$$= -\frac{4}{3}\left(\sqrt{4-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{4-x^2}\right)^5 + C$$

Question

Consider $\int \frac{1}{\sqrt{9+x^2}}\, dx$.

- Is this a $\sin(\theta)$, $\tan(\theta)$, or $\sec(\theta)$ substitution?
- What is the appropriate trig substitution?
- After the substitution what is the resulting trigonometric integral?
- What is the value of the original integral? (Video)

Question

Use an appropriate trigonometric substitution to find the area between the x-axis, $y = \frac{x^3}{\sqrt{x^2-1}}$ and $x = 2$ and $x = 3$. (Video)

Question

Consider $\int \sqrt{9 - 4x^2}\, dx$.

- Is this a $\sin(\theta)$, $\tan(\theta)$, or $\sec(\theta)$ substitution?
- What is the appropriate trig substitution?
- After the substitution what is the resulting trigonometric integral?
- What is the value of the original integral? (Video)

Guidelines

- To evaluate an integral involving $\sqrt{a^2 - x^2}$ use the trigonometric substitution $x = a\sin(\theta)$. Then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2(\theta)} \\ &= \sqrt{a^2(1 - \sin^2(\theta))} = \sqrt{a^2 \cos^2(\theta)} \\ &= a \cos(\theta)\end{aligned}$$

- To evaluate an integral involving $\sqrt{a^2 + x^2}$ use the trigonometric substitution $x = a \tan(\theta)$. Then

$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2 \tan^2(\theta)} \\ &= \sqrt{a^2(1 + \tan^2(\theta))} = \sqrt{a^2 \sec^2(\theta)} \\ &= a \sec(\theta)\end{aligned}$$

- To evaluate an integral involving $\sqrt{x^2 - a^2}$ use the trigonometric substitution $x = a \sec(\theta)$. Then

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2(\theta) - a^2} \\ &= \sqrt{a^2(\sec^2(\theta) - 1)} = \sqrt{a^2 \tan^2(\theta)} \\ &= a \tan(\theta)\end{aligned}$$

- To evaluate an integral involving $\sqrt{b^2 x^2 - a^2}$, $\sqrt{b^2 x^2 + a^2}$, or $\sqrt{a^2 - b^2 x^2}$. Then use $b\,x = a$(trig function) or $x = \frac{a}{b}$(trig function).

Homework

- IMath problems on trigonometric substitution.