Section Summary: Curves Defined by Parametric Equations

a. Definitions

parametric equations: both x and y are functions of another variable (the **parameter**) t:

$$x = f(t) \qquad y = g(t)$$

As t varies over its domain, x and y wander about in the plane, creating a **parametric curve**. As t ranges from a to b, the curve moves from **initial point** $(f(a), g(a))$ to **terminal point** $(f(b), g(b))$.

cycloid: the curve traced out by a point on the perimeter of a circle as the circle rolls along a straight line. The cycloid figures heavily in the **brachistochrone problem** (shortest time) and the **tautochrone problem** (equal time).

conchoids of Nicomedes: the family of functions given by

$$\begin{aligned} x(t) &= \quad a + \cos(t) \\ y(t) &= a\tan(t) + \sin(t) \end{aligned}$$

named "conchoids" because of their shell-like forms.

b. Summary

We now relax, and allow curves that fail the vertical line test, thinking of them as trajectories or pathways of a particle as it moves in time. Time is hence a "parameter" which locates the particle at a particular point in the plane.

We don't have to think of t as a time. For example, we might consider the following problem:

$$y(x) = x^2 + tx + 1$$

where t is a parameter. Where does the quadratic have its critical point?

$$\frac{dy}{dx} = 2x + t$$

For what value x_c is $\frac{dy}{dx} = 0$?

$$x_c = \frac{-t}{2}$$

What is the corresponding minimum?

$$y_c = y(x_c) = (\frac{-t}{2})^2 - \frac{t^2}{2} + 1 = 1 - \frac{t^2}{4}$$

As t varies the critical point varies, following the parametric curve

$$P_c = (\frac{-t}{2}, 1 - \frac{t^2}{4}) = (x(t), y(t))$$

Parametric curves are terribly important for computer graphics, as computer gamers can well imagine. As mentioned in the text, the letters in a laser printer may well be drawn as parametric curves (**Bezier curves**).

The cycloid is an important example of a parametric curve, which proves to be the solution to both the brachistochrone and the tautochrone problems.