Section Summary: 7.3: Partial Fractions

1 Mostly real thinking...

Real polynomials mean thinking about complex roots (but then kind of ignoring them!:)

a. Definitions

- **rational function**: a **ratio** of polynomials, $f(x) = \dfrac{P(x)}{Q(x)}$
- **Proper** rational expression: one for which the degree of the numerator polynomial is strictly less than that of the denominator.

b. Theorems

First of all, it's important to know that, by the **Fundamental Theorem of Algebra**, any polynomial of degree n can be written as a product of n **linear** factors:

$$P(x) = a(x - x_1)(x - x_2) \cdots (x - x_n)$$

However the x_i may be complex. If so, and if the coefficients of the polynomial are real, then complex roots appear only as complex pairs: $u \pm vi$ (where u and v are real numbers, and the imaginary number i is the square root of -1).

If you multiply the two linear terms corresponding to such a pair, $(x - (u + vi))$ and $(x - (u - vi))$, you get

$$(x - (u + vi))(x - (u - vi)) = x^2 - ((u + vi) + (u - vi))x + (u + vi)(u - vi)$$

which works out to

$$(x - (u + vi))(x - (u - vi)) = x^2 - 2ux + (u^2 + v^2)$$

(that is, a quadratic, with real coefficients).

So the upshot is that every real polynomial can be written as a product of linear terms and quadratic terms (with complex roots), all with real coefficients.

c. Properties/Tricks/Hints/Etc.

Every rational function can be expressed as a sum of a polynomial and a proper rational expression. This is the most illuminating way to write the rational function, because it shows off the behavior of the rational function. If f is improper, then we can rewrite it as

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials, with $deg(R) < deg(Q)$.

We can find this representation by long-division (which you may or may not recall, but it's not difficult).

Now as $x \to \infty$, $\dfrac{R(x)}{Q(x)} \to 0$, since the degree of Q is greater than the degree of R. This means that, far from the origin, $f(x) \approx S(x)$. That is, f looks like the polynomial S.

So near the origin, the proper rational function $\dfrac{R(x)}{Q(x)}$ may dominate – for example, if Q has real roots, this expression may blow up (or down) to infinity (or negative infinity). This is the very un-polynomial-like behavior, which is characteristic of the nastier rational functions.

d. Summary

Now what we learn in this section is that

$$\int f(x)dx = \int \left(S(x) + \frac{R(x)}{Q(x)}\right) dx = \int (S(x)dx + \int \frac{R(x)}{Q(x)}dx$$

The first part is easy (integral of a polynomial), whereas the second part can be rewritten as a sum of "fractional" terms:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(x-x_1)} + \frac{A_2}{(x-x_2)} + \ldots + \frac{A_n}{(x-x_n)}$$

if the roots of Q are distinct and real.

If the roots repeat, or if the roots are complex, we need to adjust things a little.

If we have repeated real roots (say x_1 repeats 3 times), then we'll have terms

$$\frac{A}{(x-x_1)}, \frac{B}{(x-x_1)^2}, \frac{C}{(x-x_1)^3}$$

For each quadratic term having complex roots, non-repeated, ax^2+bx+c, there is a term

$$\frac{Ax+B}{ax^2+bx+c}$$

Once again, if this term repeats in the factorization of Q, then we need one such term for each power of the quadratic in the partial fraction decomposition.

2 If you'll deal with "complexities"

Consider

$$\int \frac{1}{x^2+1}dx = \tan^{-1}(x) + C$$

We know that $x^2+1 = (x+i)(x-i)$, so

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i}$$

where A and B are complex numbers.

Multiplying out, we get

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{A(x-i)+B(x+i)}{(x+i)(x-i)} = \frac{A(x-i)+B(x+i)}{x^2+1}$$

Since the first and last expressions are equal, their numerators must be equal. So

$$1 = A(x-i) + B(x+i) = (A+B)x + (B-A)i$$

Since the left-hand side is independent of x, the right-hand side must be independent of x, too. Hence

$$A + B = 0 \Longrightarrow B = -A$$

Thus $(B-A)i = -2Ai = 1$, so $A = \frac{i}{2}$. Therefore $B = \frac{-i}{2}$, and

$$\frac{1}{x^2+1} = \frac{i}{2}\left(\frac{1}{x+i} - \frac{1}{x-i}\right)$$

Therefore

$$\int \frac{1}{x^2+1}dx = \int \frac{i}{2}\left(\frac{1}{x+i} - \frac{1}{x-i}\right)dx = \frac{i}{2}\left(\ln(x+i) - \ln(x-i)\right) + C$$

Now you might be wondering about several things at this point. Number one, you're wondering whether you ever should have thought that you'll deal with "complexities". :)

Seriously, you might be wondering what to make of those logs (and wonder about the missing absolute values, etc.). There are several issues, wrapped up with these "complexities".

In particular, a complex number can be represented by a product of a positive real number and a complex exponential (which can itself be represented as a complex sum of a sine and cosine! Miracles, it seems....):

$$x + i = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$$

The positive real number r is called the "modulus" (the size of the complex number), and given in this case by $r = \sqrt{x^2+1}$. Once again we equate real and imaginary parts, to determine that

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2+1}}$$

and

$$\sin(\theta) = \frac{1}{r} = \frac{1}{\sqrt{x^2+1}}$$

Furthermore, we can use our properties of logs to write

$$\ln(x+i) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta$$

and it turns out that

$$\ln(x-i) = \ln\left(re^{-i\theta}\right) = \ln(r) - i\theta$$

so that

$$\int \frac{1}{x^2+1}dx = \frac{i}{2}\left(\ln(r) + i\theta - (\ln(r) - i\theta)\right) + C = \frac{i}{2}(2i\theta) + C = -\theta + C$$

From our definitions of $\cos(\theta)$ and $\sin(\theta)$, we can see that $\tan(\theta) = \frac{1}{x}$, so that

$$-\theta = -\tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{-1}{x}\right)$$

That's interesting! So it turns out that

$$\int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C = \tan^{-1}\left(\frac{-1}{x}\right) + C$$

Well, sort of...! Plot both of those arctans, and you'll see something very interesting! One of those arctans has a very serious problem when it comes to $x = 0$. But notice, in particular, that

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{-1}{x}\right)\right) = \frac{1}{x^2+1}$$

everywhere but at $x = 0$.

The mysteries that you discover should lead you to take complex analysis, in order to resolve them with oddities called "branch cuts", and "singularities", and the like....