# Taylor Series

MAT 229, Spring 2021

Week 13

## Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 11.10: Taylor Series
- Boelkins/Austin/Schlicker's Active Calculus
  Section 8.5: Taylor Polynomials and Taylor Series

## Functions as power series

### Questions

Let $k(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- What is the domain of this function?
- What is $k(0)$?
- Write out the first 5 terms of $k(x)$, the first 5 terms of $k'(x)$ and the first 5 terms of $\int k(x)\,dx$. What well known function is this?

(Video)

## Taylor series

Given a function we want to find a power representation for it if possible.

- We need to specify a center of convergence (a **MacLaurin series** is just a **Taylor series centered at 0**).
- A good power representation is one for which it converges for more than just the center.
- Building a power series from a known series, like the geometric series, works in some but not all cases.

## Questions

Given function $f(x)$, suppose it has a power series representation centered at $x = a$.

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

What are the values $c_n$?

- Write this function with the first 6 terms of the power series written out ($n = 0, 1, 2, 3, 4, 5$).
- What happens when $x = a$ is plugged into this function?
- What happens when $x = a$ is plugged into this function after both sides are differentiated.
- What happens when $x = a$ is plugged into this function after both sides are differentiated twice.
- What happens when $x = a$ is plugged into this function after both sides are differentiated three times.
- What happens when $x = a$ is plugged into this function after both sides are differentiated four times.
- In general, what appears to be true?

(Video)

## Definition

If $f(x)$ is differentiable to all orders at $x = a$, then its *Taylor series* centered at $a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

From the above, if $f(x)$ has a power series representation centered at $a$, it must be its Taylor series.

## Comment:

- The **tangent line** (the *osculating* line) to the graph at $x = a$ is best written -- and best thought of, perhaps -- as $S_1$. Write that out, and confirm it.
- What would you call $S_2$?
- What would you call $S_3$, etc.?

## Questions

- What is the Taylor series centered at 0 for $f(x) = \frac{1}{1-x}$?
- What is $f^{(100)}(0)$?
- What is $f^{(501)}(0)$?

(Video)

## Questions

- What is the Taylor series centered at 0 for $f(x) = \ln(1 + x)$?
- What is $f^{(100)}(0)$?

- What is $f^{(501)}(0)$?
- What is the Taylor series centered at 0 for $x \ln(1+x)$?

(Video)

## Questions

Let $f(x) = e^x$.

- What is the Taylor series for $f(x)$ centered at 0?

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Series[Exp[x], {x, 0, 10}]
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  - What is $f^{(n)}(x)$?
  - What is $f^{(n)}(0)$?
- For which values of $x$ does this series converge?
- Using the above, what is the Taylor series for $k(x) = x\,e^x$ centered at 0?
- What is true about $k^{(n)}(0)$?

(Video)

## Questions

Let $g(x) = \sin(x)$.

- What is the Taylor series for $g(x)$ centered at 0?
  - What is $g^{(n)}(x)$?
  - What is $g^{(n)}(0)$?
- For which values of $x$ does this series converge?
- Using the above, what is the Taylor series for $m(x) = x^2 \sin(x)$ centered at 0?
- What is true about $m^{(n)}(0)$?

(Video)

## Questions

Let $h(x) = \cos(x)$.

- What is the Taylor series for $h(x)$ centered at 0?
- For which values of $x$ does this series converge?
- What is $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$?
- Using the above, what is the Taylor series centered at 0 for $\frac{1-\cos(x)}{x^2}$?

(Video)

### Questions

Let $h(x) = \cos(x)$.

- What is the Taylor series for $h(x)$ centered at $\pi/2$?
- For which values of $x$ does this series converge?

(Video)

### Questions

Let $k(x) = \cos(x^2)$.

- Using the Taylor series centered at 0 for $\cos(x)$, what is the Taylor series centered at 0 for $k(x)$?
- The derivative $k^{(n)}(0)$ is 0 for which values of $n$?

(Video)

# Homework

- IMath problems on Taylor Series.