Lines and Planes

MAT 229, Spring 2021

Week 16

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 12.5: Lines and planes
- Boelkins/Austin/Schlicker's *Active Multivariable Calculus*
  Section 9.5: Lines and planes

Review

Questions

- How can we easily determine if two vectors are perpendicular or not?
- What is a simple way to find a 3D vector that is orthogonal (perpendicular) to two given vectors?

(Video)

Questions

Consider the parametric equations

$x(t) = 3t + 5$

$y(t) = -2t + 1$

- What is the point associated with $t = 0$?
- What is the point associated with $t = 1$?
- What is the vector with initial point $(x(0), y(0))$ and terminal point $(x(1), y(1))$?
- Find a Cartesian equation for this parametrically defined curve.

(Video)

Questions

Consider the parametric equations

$x(t) = a\,t + x_0$

$y(t) = b\,t + y_0$

- What is the point associated with $t = 0$?
- What is the point associated with $t = 1$?
- What is the vector with initial point $(x(0), y(0))$ and terminal point $(x(1), y(1))$?
- Find a Cartesian equation for this parametrically defined curve to show it is a line.
- What is its slope?
- How does the vector found above compare with the line?

(Video)

Parametric equations of lines in space

Questions

- What information is needed to find an equation for a line in the plane?
- How many points determine a line in the plane?
- How many points determine a line in space?

(Video)

Parametric equations

The parametric equations

$x(t) = a\,t + x_0$

$y(t) = b\,t + y_0$

$z(t) = c\,t + z_0$

produce a line in space.

As a vector equation:

It's better to think of this as a vector equation:

$$\langle x(t), y(t), z(t)\rangle = \langle a\,t + x_0, b\,t + y_0, c\,t + z_0\rangle = \langle a, b, c\rangle\,\mathrm{t} + \langle x_0, y_0, z_0\rangle$$

Furthermore, we can make the analogy of the vector $\langle a, b, c\rangle$ as "the slope", and $\langle x_0, y_0, z_0\rangle$ as "the intercept" -- the point you're at when $t=0$.

Two-dimensional version:

Think about the two-dimensional analogue: in 2d, we think of x as the independent and y as the dependent variable. The two corresponding parametric equations are

$x(t) = t$

$y(t) = m\,t + y_0$

which produce a line in the plane:

$$\langle x(t), y(t)\rangle = \langle t, m\,t + y_0\rangle = \langle 1, m\rangle\, t + \langle 0, y_0\rangle$$

Questions

Consider the line given by

$x(t) = 2\,t + 3$

$y(t) = -t + 2$

$z(t) = 3\,t$

- What is the point $(x(0), y(0), z(0))$?
- What is the point $(x(1), y(1), z(1))$?
- What is the vector that points from $(x(0), y(0), z(0))$ to $(x(1), y(1), z(1))$?
- What relation does this vector have to the line?

(Video)

Questions

Given the parametric equations

$x(t) = a\,t + x_0$

$y(t) = b\,t + y_0$

$z(t) = c\,t + z_0$

- What is the point on the line associated with $t = 0$?
- What is a vector parallel to the line?

(Video)

Questions

- Find parametric equations for the line through point $(-1, 0, 1)$ and parallel to vector $\langle 3, 4, 0\rangle$. (Video)
- Find parametric equations for the line through point $(2, 3, -1)$ and parallel to the line given parametrically by $x = -t$, $y = 2\,t - 3$, $z = t + 1$. (Video)
- Find parametric equations for the line through points $(0, 1, 1)$ and $(2, 3, 5)$.
- Find a different set of parametric equations for the line through points $(0, 1, 1)$ and $(2, 3, 5)$.

(Video)

Vector equation of a line in space

Definitions

- The *position vector* $\vec{u} = \langle a, b, c \rangle$ is the vector with initial point at the origin and terminal point at (a, b, c).
- The *parametric vector equation*

  $$\vec{r}(t) = t\,\vec{u} + \vec{r}_0$$

  for each value of t is the position vector for a point on the line parallel to vector $\vec{u}$, that passes through the terminal point of position vector $\vec{r}_0$.

Question

- What is a vector equation for the line that passes through $(1, 0, 2)$ and that is parallel to vector $\langle -1, 1, 1 \rangle$? (Video)

Planes

Question

- What is the minimum information needed to determine a plane? (Video)

Questions

Given a plane let

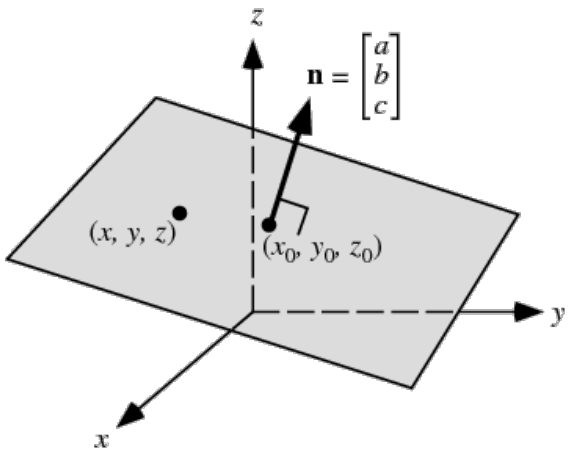
$\vec{n}$ be a vector perpendicular to it,

$P_0(x_0, y_0, z_0)$ be a particular point on it.

If $P(x, y, z)$ is any point on the plane

- What is true about the vector $\overrightarrow{P_0P}$?
- What relation does this vector have with vector $\vec{n}$?
- If $\vec{n}$ in component form is $\langle a, b, c \rangle$, write an equation that x, y, z must satisfy to lie on the plane that passes through (x_0, y_0, z_0) and that is perpendicular (normal) to $\vec{n}$.

(Video)



Equation of a plane

The points with coordinates (x, y, z) that satisfy the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

form a plane that passes through (x_0, y_0, z_0) and that is normal to vector $\langle a, b, c\rangle$.

Frequently we solve for z, and so think of $z(x,y)$ as the plane, a function of the two "independent" coordinates x and y:

$$z(x, y) = [a(x - x_0) + b(y - y_0) + c(-z_0)]/(-c)$$

Or, maybe better yet, in analogy with the equation of a line in two dimensions

$$z = z_0 - \left[\frac{a}{c}(x - x_0) + \frac{b}{c}(y - y_0)\right] = z_0 + m_x(x - x_0) + m_y(y - y_0).$$

And so we defined “slopes” in both the x and y directions.

Alternatively (certainly better),

$$\langle a, b, c\rangle \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) = 0$$

That is, the vector $\langle a, b, c\rangle$ and a vector lying in the plane, given as the difference of the two vectors $\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle$, has a dot product of 0: they're perpendicular.

Questions

- Find an equation for the plane that passes through point $(1, 0, -1)$ and that is normal to vector $\langle 2, 3, -4\rangle$. (Video)
- What is a vector normal to the plane given by $3(x + 2) - y + 2(z - 1) = 0$? What is a point on this plane? (Video)
- Rewrite the last equation so that all the variables are on one side of the equation and the constants are summed on the other side. (Video)
- What is an equation for the plane that passes through $(0, 1, 0)$ that is parallel to both $\langle 1, 1, 1\rangle$ and $\langle 0, 1, -1\rangle$? (Video)
- Find an equation for the plane that passes through the three points $P_0(1, 1, 1)$, $P_1(2, 0, -1)$, and $P_2(0, 1, 3)$. (Video)

- Find an equation for the plane that is parallel to $x + 2y + 3z = 20$ and that passes through the point $(1, 0, -1)$. (Video)

Homework

- IMath problems on lines and planes.