Exponential Functions

Supporting materials

If you wish to get a different perspective on the notes below try either of the following textbook sections.

- Stewart's *Calculus*
  Section 6.2: Exponential function and their derivatives
- Boelkins/Austin/Schlicker's Active Calculus
  Section 2.6: Elementary derivative rules

Overview

An exponential function has the form $f(x) = a(b)^x$ where a and b are some constants. The number b is called the *base*.

Examples

- Social media: you develop the latest and greatest social media app. If you introduce it to 2 people, both of whom introduce it to 5 new people after one week, each of whom introduces it to 5 new people after week two, etc., how many people are using it after 10 weeks?
- Disease: the spread of disease can often be modeled using exponential functions. It is similar to the social media effect.
- Forensic science: heat loss can be reasonably modeled using exponential functions. In particular, if the ambient temperature stays mostly constant, then the temperature T of a body that is cooling at time t is approximated by the function $T(t) = a + b \cdot c^t$ where a is the ambient temperature with b and c being constants that depend on the original temperature of the body and how fast it loses heat. One tool in the CSI toolbox for determining time of death is based on this model.
- Finance: determining the value of an investment is based on exponential functions.

Question:

Using a graphing calculator, or a plotting website like Desmos, graph $y = b^x$ for various values of b. Form some conjectures about what is true for different values of b. (Video)

Question

What are equivalent expressions for

- b^0
- b^{-1}
- b^{x+y}
- b^{x-y}
- $(b^x)^y$
- $b^{1/n}$

(Video)

Derivatives of exponential functions

The graph of $y = 2^x$ is

```
Plot[2^x, {x, -3, 3}]
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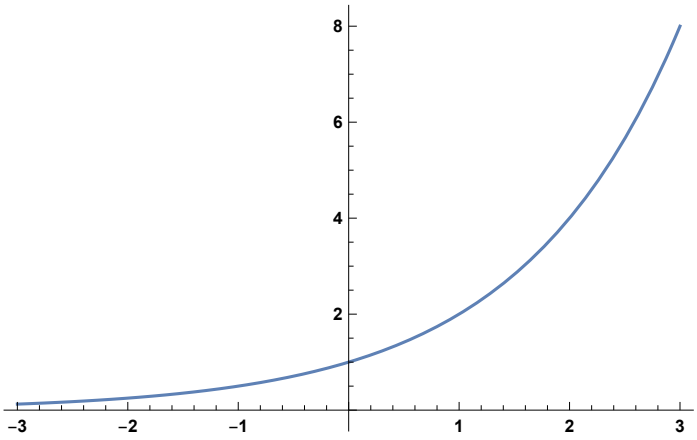

Questions

- Does the power rule apply to the derivative of 2^x? Why or why not?
- What is the limit definition of the derivative for any function $f(x)$?

Derivative definition

Whenever you encounter a function whose derivative you do not know, you can always go back to the limit definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Apply this to $f(x) = 2^x$.

$$\frac{d}{dx}(2^x) = \lim_{h\to 0} \frac{2^{x+h}-2^x}{h}$$

Using the exponent property that $2^{x+h} = 2^x * 2^h$, this derivative becomes

$$\frac{d}{dx}(2^x) = \lim_{h\to 0} \frac{2^x(2^h)-2^x}{h}$$
$$= \lim_{h\to 0} 2^x \frac{2^h-1}{h}$$
$$= 2^x \lim_{h\to 0} \frac{2^h-1}{h}$$

The limit

$$\lim_{h\to 0} \frac{2^h-1}{h}$$

is just some number. Numerically evaluate it by plugging in values of h that get successively closer to 0.

h	$\frac{2^h-1}{h}$
0.1	$\frac{2^{0.1}-1}{0.1} \approx 0.717735$
0.01	$\frac{2^{0.01}-1}{0.01} \approx 0.695555$
0.001	$\frac{2^{0.001}-1}{0.001} \approx 0.693387$
0.0001	$\frac{2^{0.0001}-1}{0.0001} \approx 0.693171$
0.00001	$\frac{2^{0.00001}-1}{0.00001} \approx 0.693150$
0.000001	$\frac{2^{0.000001}-1}{0.000001} \approx 0.693147$

The limit does appear to exist and tends to some number that to three decimal places is 0.693.

This means the derivative of 2^x is equal to itself (2^x) times that number.

Question

- Apply the derivative definition to get $f'(x)$ if $f(x) = 3^x$. What is the corresponding number for this exponential function? (Video)

Natural exponential function

There exists a number between 2 and 3, Euler's number e

such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$ so that $\frac{d}{dx}(e^x) = e^x$.

To 20 decimal places it looks like

```
N[E, 20]
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2.7182818284590452354
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Questions

- What is the derivative of $5x^2 + \cos(3x) - 4e^x$? (Video)
- Find an equation for the tangent line to $y = e^{2x}$ at $x = 0$. (Video)
- What is the absolute maximum and the absolute minimum values of $g(x) = 3e^{-x^2-6x}$ for $-5 \le x \le 5$? (Video)
- Compute the area of the region bounded by $y = e^x$ and $y = x$ for $0 \le x \le 2$. (Video)

- What is the volume of the solid of revolution obtained by rotating $y = e^x$ about the x-axis for $-1 \leq x \leq 1$? (Video)

Homework

- IMath problems on exponential functions.