Derivatives of Logarithm Functions

Supporting materials

If you wish to get a different perspective on the notes below try either of the following textbook sections.

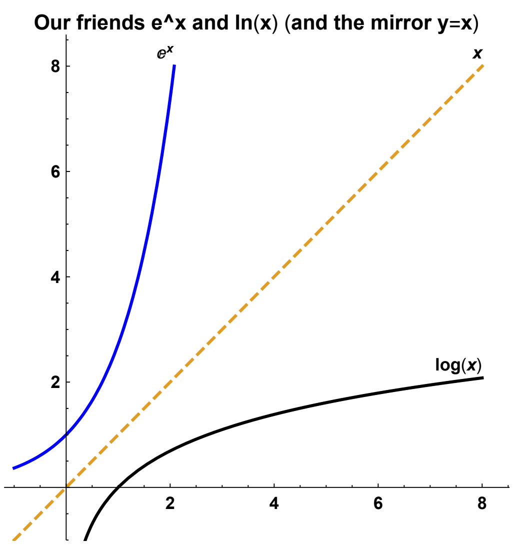
- Stewart's *Calculus*
  Section 6.4: Derivatives of logarithmic functions
- Boelkins/Austin/Schlicker's Active Calculus
  Section 2.6: Derivatives of Inverse Functions

Review

Logarithms

Since $f(x) = e^x$ is a one-to-one function, it has an inverse function. That function is called the *natural logarithm* function and is denoted

$$f^{-1}(x) = \ln(x)$$



Everything we know about $\ln(x)$ comes from the fact that

$$y = \ln(x) \longleftrightarrow x = e^y$$

These functions are "reflections of each other", across the mirror of y=x.

In general $f(x) = a^x$ is a one-to-one function as long as base a is a positive number other than 1. Since it is one-to-one, it has an inverse and that inverse is denoted

$f^{-1}(x) = \log_a(x)$

This means $y = \log_a(x)$ and $x = a^y$ are **equivalent**.

You can translate any log problem into an exponential problem, and any exponential problem into an equivalent log problem.

Logarithm/Exponential mirrored properties:

- $a^x\, a^y = a^{x+y} \longleftrightarrow \log_a(u\, v) = \log_a(u) + \log_a(v)$
- $\frac{a^x}{a^y} = a^{x-y} \longleftrightarrow \log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)$
- $(a^x)^y = a^{x\,y} \longleftrightarrow \log_a(u^r) = r \log_a(u)$
- $a^{\log_a(x)} = x \longleftrightarrow a^x = e^{\ln(a^x)} = e^{x \ln(a)}$

Questions

Let $g(x) = 2\, e^x - 5\, x$.

- Find all critical numbers for $g(x)$. Classify each as either a local maximum, a local minimum, or neither.
- Determine where $y = g(x)$ is concave up and where it is concave down.

Question

Let $h(x) = x\, e^{x/3}$.

- Find the absolute maximum value and the absolute minimum value of $h(x)$ for $-4 \le x \le 4$. (Video)

Question

- What is an equivalent way to write $g(x) = (x^2 + 1)^{\cos(x)}$? What is its domain? What is its derivative? (Video)

Derivative of ln(x)

Let $y = \ln(x)$, then $x = e^y$. Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$$

$$\longrightarrow 1 = e^y \frac{dy}{dx}$$

$$\longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

The derivative of $\ln(x)$ is

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} = x^{-1}$$

Wow! That's a strange and interesting collision, between the natural log function *ln* and powers of x. If you recall, there was one power that didn't fit the power rule of integration,

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \quad :$$

and that ' s the integrand x^{-1}. The anti-derivative of every other power is a power -- there's just this one that didn't work. What function would serve as its anti-derivative? A log!

Questions

- Find any maximum or minimum points to $f(x) = x \ln(x)$. (Video)
- What is an equation for the tangent line to $g(x) = x^{\sin(x)}$ when $x = \pi/2$? (Video)
- Let $h(x) = \ln(-x)$. What is its domain? What is its derivative? (Video)
- What is $\int x^{-1} \, dx$? (Video)
- Find the area of the region bounded by the x-axis and $y = \frac{1}{x}$ for $1 \le x \le 2$. (Video)
- Find the area of the region bounded by the x-axis and $y = \frac{1}{x}$ for $-3 \le x \le -1$. (Video)
- Find the volume of the solid of revolution obtained by rotating the region bounded by the x-axis and $y = 1 + \frac{1}{x}$ for $1/2 \le x \le 2$. (Video)

Homework

- Weekly assignment 2
- IMath problems on derivatives of logarithmic functions