Indeterminate Forms

Supporting materials

If you wish to get a different perspective on the notes below try either of the following textbook sections.

- Stewart's *Calculus*

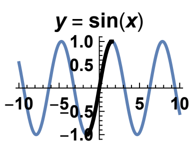
  Section 6.8 : Indeterminate Forms and *L*' Hopital' *s* Rule

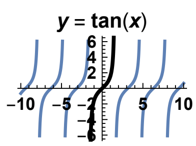
- Boelkins/Austin/Schlicker's Active Calculus

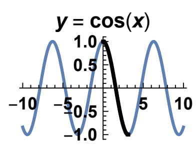
  Section 2.8: Using Derivatives to Evaluate Limits

Review

Trigonometric functions, and their inverses:







Questions

- What is the domain and range of $\sin^{-1}(x)$? Of $\tan^{-1}(x)$? Of $\cos^{-1}(x)$?
- When is an "identity" an identity?
  - What is $\tan\left(\tan^{-1}(0.3)\right)$? What is $\tan^{-1}(\tan(\pi/4+2\pi))$?
  - What is $\cos^{-1}(\cos(\pi/5))$? What is $\cos^{-1}(\cos(-\pi/5))$?
  - What is $\sin\left(\cos^{-1}(1/2)\right)$?

Right triangles

To find $\tan(\sin^{-1}(0.4))$, let $\theta = \sin^{-1}(0.4)$ so that $0.4 = \sin(\theta)$. Represent $\sin(\theta)$ in a right triangle.

Using this triangle, $\tan(\sin^{-1}(0.4)) = \tan(\theta) = \frac{0.4}{\sqrt{1-0.4^2}} = \frac{0.4}{\sqrt{0.84}} \approx 0.436$.

Questions

Use a right triangle to find a formula for $\sec(\tan^{-1}(x))$.

Indeterminate forms

End behavior of functions requires that we deal with limits. An “end” may occur where $x \to \pm\infty$, or it may occur internally -- for instance, where there is a vertical asymptote.

Questions

- What is $\lim_{x\to 0} \frac{2x+1}{x-1}$?
- What is $\lim_{x\to 0} \frac{4x+1}{x}$?
- What is $\lim_{x\to 0} \frac{2x}{x}$?
- What is $\lim_{x\to 0} \frac{x}{5x}$?
- What is $\lim_{x\to \infty} \frac{x}{2}$?
- What is $\lim_{x\to \infty} \frac{1}{x}$?
- What is $\lim_{x\to -\infty} \frac{x-1}{x+1}$?
- What is $\lim_{x\to \infty} \frac{x^2}{x}$?

Given a limit $\lim_{x\to a} f(x)$, if we can simply evaluate $f(a)$ as the limit we say $\lim_{x\to a} f(x)$ is *determinate.* If we cannot simply evaluate $f(x)$ at $x = a$, we say the limit is *indeterminate* -- it may or may not exist. Perhaps some algebra will help!

Indeterminate limits

- $\frac{0}{0}$, a small number divided by a small number -- hmm, could be anything. More work is needed.
- $\frac{\infty}{\infty}$, a large number divided by a large number could be anything. More work is needed.

The most important indeterminate form in calculus is undoubtedly the limit definition of the derivative, in either of its forms:

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
or
$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

In the limit as $h\to 0$, the numerator goes to 0 and the denominator goes to 0; similarly as $x\to a$ in the second form of the limit definition.

L’Hôpital’s rule

If you have a limit of a quotient which is either a $\frac{0}{0}$ or an $\frac{\infty}{\infty}$ limit, then the following is true if the limit (and the derivatives) exists:

$$\lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{g'(x)}{h'(x)}$$

Warning: If the limit is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the above two limits are not equal.

Example

To evaluate $\lim_{x\to\infty} \frac{x}{e^x}$, first notice that plugging in ∞ for x produces $\frac{\infty}{e^\infty} = \frac{\infty}{\infty}$. We can use L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Questions

Evaluate the following limits.

- $\lim_{x\to\infty} \frac{x}{\ln(x)}$(Video)
- $\lim_{x\to\infty} \frac{\tan^{-1}(x)}{x^2+1}$(Video)
- $\lim_{x\to 1} \frac{x^2-3x+2}{\sin(x-1)}$ (Video)
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$(Video)

Why it works

For the $\frac{0}{0}$ case, this means $f(a) = 0$ and $g(a) = 0$. Remember the limit definition of the derivative

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

Since $f(a) = 0 = g(a)$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}$$

Questions

Can we use L'Hôpital's rule on $\lim_{x\to 1} \frac{x-1}{e^{x-1}}$? Compare the actual value of this limit with the limit that comes from L'Hôpital's rule. (Video)

Other indeterminate forms

There are other indeterminate forms. Sometimes we can use L'Hôpital's rule to evaluate them *if* we can rewrite into either the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Other forms

- $\infty - \infty$ or a large number minus a large number.

- $0 \cdot \infty$ or a number close to zero times a large number.
- Indeterminate powers
  - 0^0 or a small number raised to another small number.
  - ∞^0 or a large number raised to a small number.
  - 1^∞ or a number close to 1 raised to a large power.

Difference example

Evaluate $\lim_{x\to\infty}(\ln(2x+1)-\ln(3x-5))$.

Use log properties to rewrite as a fraction.

$\lim_{x\to\infty}(\ln(2x+1)-\ln(3x-5))$

$= \lim_{x\to\infty}\ln\left(\frac{2x+1}{3x-5}\right)$

$= \ln\left(\lim_{x\to\infty}\frac{2x+1}{3x-5}\right)$ -- (What allows us to do this?)

$= ?$

(Video)

Product example

Evaluate $\lim_{x\to\infty} x(\pi/2-\tan^{-1}(x))$.

Rewrite one of the factors as a fraction, factor $= \frac{1}{1/\text{factor}}$.

$\lim_{x\to\infty} x(\pi/2-\tan^{-1}(x)) = \lim_{x\to\infty}\frac{(\pi/2-\tan^{-1}(x))}{1/x} = ?$

(Video)

Questions

- $\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right)$ (Video)
- $\lim_{x\to 0^+} x\ln(x)$ (Video)

Power example

A limit that comes from finance is

$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x$

Change the exponential expression to base e and take the limit of the exponent.

$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{r}{x}\right)^x\right)} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{r}{x}\right)}$

$= e^{\lim_{x\to\infty} x\ln(1+r/x)} = ?$

Questions

Consider the function $g(x) = (x+e^x)^{1/x}$. It is defined on $(0,\infty)$.

- What is $\lim_{x \to 0^+} g(x)$?
- What is $\lim_{x \to \infty} g(x)$?

Homework

- IMath problems on indeterminate forms.
- Weekly Assignment #2 is also due.
- First exam is coming up. You’ve got a lot of IMath sets due this week. I hope that those are going fine. You can use late passes, if you must.