Trigonometric Integrals

Review

We have (at least) three antidifferentiation techniques at our disposal:

1. Recognize it immediately as the derivative of some function, for example
   $$\int e^x\,dx = e^x + C$$
   (i.e. stare at it until the answer comes to you -- the most general method!).

2. Try substituting for some internal part. Don't forget the differential. For example, to evaluate $\int x^2 \cos(x^3)\,dx$ use the substitution $u = x^3$ and $du = 3\,x^2\,dx$ to replace the above integral with the simpler one
   $$\int \frac{1}{3}\cos(u)\,du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3) + C$$
   If you have limits (i.e. a definite integral), make sure that you clarify the variable to which the limits apply. In particular, don't substitute to a variable u, only to use limits appropriate to the original x....

3. Integration by parts:
   $$\int u\,dv = u\,v - \int v\,du$$
   or, as given in my preferred form,
   $$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Questions

Choose an appropriate technique to evaluate the following integrals.

- Find the area of the region bounded above by $y = x\,e^{-x}$, below by the x-axis, and the line $x = 5$.
- Find the volume of the surface of revolution generated by rotating about the x-axis the region bounded above by $y = x\sec(x^3)$, below by the x-axis, and the line $x = 1$.

Trigonometric identities

Pythagorean identities

$\cos^2(\theta) + \sin^2(\theta) = 1$

Questions

- Divide this equation by $\cos^2(\theta)$ to get another Pythagorean identity.
- Divide this equation by $\sin^2(\theta)$ to get another Pythagorean identity.

Double angle formulas

The sine and cosine sum formulas are very important:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \,.$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Questions

- What is $\sin(2\,\theta) = \sin(\theta+\theta)$?
- What is $\cos(2\,\theta) = \cos(\theta+\theta)$?
- Rewrite this using a Pythagorean trigonometric identity so that $\cos(2\,\theta)$ is in terms of only $\cos(\theta)$.
- Rewrite this using a Pythagorean trigonometric identity so that $\cos(2\,\theta)$ is in terms of only $\sin(\theta)$.

Trigonometric integrals

Questions

What are the following derivatives?

- $\frac{d}{dx}\tan(x)$
- $\frac{d}{dx}\sec(x)$
- $\frac{d}{dx}\cot(x)$
- $\frac{d}{dx}\csc(x)$

Questions

- Evaluate $\int \cos^4(x)\sin(x)\,dx$ using the substitution $u = \cos(x)$.
- Evaluate $\int \sin^6(x)\cos^3(x)\,dx$ by writing this as $\int \sin^6(x)\cos^2(x)\cos(x)\,dx = \int \sin^6(x)\left(1-\sin^2(x)\right)\cos(x)\,dx$.
- Find the area under the curve $y = \sin^3(x)\cos^3(x)$ for $0 \le x \le \pi/2$.
- Evaluate $\int \cot^4(x)\csc^2(x)\,dx$.
- Evaluate $\int \tan^2(x)\sec^4(x)\,dx$.
- Since $\cos(2\,\theta) = 2\cos^2(\theta) - 1$, solve for $\cos^2(\theta)$ to get $\cos^2(\theta) = \frac{1+\cos(2\,\theta)}{2}$. Use this to evaluate

  $$\int \cos^2(\theta)\,d\theta = \int \frac{1+\cos(2\,\theta)}{2}\,d\theta.$$
- Find the volume of the solid of revolution obtained by rotating about the x-axis the region above the x-axis and one arch of the sine curve $y = \sin(x)$.