Trigonometric Substitution

Summary

- To evaluate an integral involving $\sqrt{a^2 - x^2}$ use the trigonometric substitution $x = a\sin(\theta)$. Then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2(\theta)} \\ &= \sqrt{a^2(1 - \sin^2(\theta))} = \sqrt{a^2\cos^2(\theta)} \\ &= |\ a\cos(\theta)\ |\end{aligned}$$

- To evaluate an integral involving $\sqrt{a^2 + x^2}$ use the trigonometric substitution $x = a\tan(\theta)$. Then

$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2\tan^2(\theta)} \\ &= \sqrt{a^2(1 + \tan^2(\theta))} = \sqrt{a^2\sec^2(\theta)} \\ &= |\ a\sec(\theta)\ |\end{aligned}$$

- To evaluate an integral involving $\sqrt{x^2 - a^2}$ use the trigonometric substitution $x = a\sec(\theta)$. Then

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2\sec^2(\theta) - a^2} \\ &= \sqrt{a^2(\sec^2(\theta) - 1)} = \sqrt{a^2\tan^2(\theta)} \\ &= |\ a\tan(\theta)\ |\end{aligned}$$

- To evaluate an integral involving $\sqrt{b^2x^2 - a^2}$, $\sqrt{b^2x^2 + a^2}$, or $\sqrt{a^2 - b^2x^2}$. Then use $b\,x = a$(trig function) or $x = \frac{a}{b}$(trig function).

Example

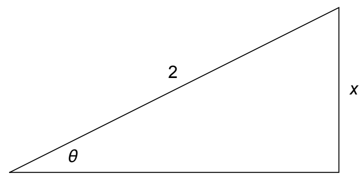
To evaluate $\int x^3\sqrt{4 - x^2}\,dx$ use the substitution $x = 2\sin(\theta)$ with $-\pi/2 \le \theta \le \pi/2$. That means $dx = 2\cos(\theta)\,d\theta$.

$$\begin{aligned}\text{I} = \int x^3\sqrt{4 - x^2}\,dx &= \int 8\sin^3(\theta)\sqrt{4 - 4\sin^2(\theta)}\;2\cos(\theta)\,d\theta \\ &= \int 8\sin^3(\theta)\sqrt{4\cos^2(\theta)}\;2\cos(\theta)\,d\theta \\ &= \int 8\sin^3(\theta)\,2\cos(\theta)\,2\cos(\theta)\,d\theta \\ &= 32\int \sin^3(\theta)\cos^2(\theta)\,d\theta \\ &= 32\int \sin^2(\theta)\cos^2(\theta)\sin(\theta)\,d\theta \\ &= 32\int (1 - \cos^2(\theta))\cos^2(\theta)\sin(\theta)\,d\theta\end{aligned}$$

Now make the substitution $u = \cos(\theta)$ so that $du = -\sin(\theta)\,d\theta$ or $\sin(\theta)\,d\theta = -du$.

$$= -32 \int (1-u^2)\, u^2 \, du$$
$$= -32 \int (u^2 - u^4)\, du$$
$$= -32 \left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C$$
$$= -32 \left(\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5} \right) + C$$

Since $x = 2\sin(\theta)$, $\sin(\theta) = x/2$:



From the corresponding right triangle, $\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$. The final answer is

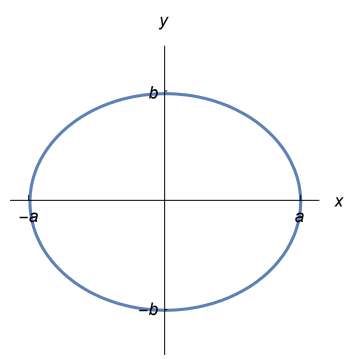
$$-32 \left(\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5} \right) + C = -32 \left(\frac{\left(\sqrt{4-x^2}/2\right)^3}{3} - \frac{\left(\sqrt{4-x^2}/2\right)^5}{5} \right) + C$$
$$I = -\frac{4}{3} \left(\sqrt{4-x^2} \right)^3 + \frac{1}{5} \left(\sqrt{4-x^2} \right)^5 + C$$

Problems to consider:

1. Use trigonometric substitution to evaluate $\int \sqrt{4 - 9x^2}\, dx$.

2. Use trigonometric substitution to evaluate $\int_0^2 \frac{x^3}{\sqrt{2x^2+1}}\, dx$.

3. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:



3.1. Solve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y.

3.2. Find the area of this ellipse.

4. Trigonometric substitution can be useful even if no square roots are involved. Use it to evaluate $\int \frac{1}{(1+x^2)^2}\, dx$.

5. What is an appropriate trigonometric substitution to use in the integral $\int \sqrt{(2x-1)^2 + 25}\, dx$? You don't have to evaluate the integral, just provide the substitution along with reasons for your choice.