

The Origins of Greek Mathematics¹



Though the Greeks certainly borrowed from other civilizations, they built a culture and civilization on their own which is

- The most impressive of all civilizations,
- The most influential in Western culture,
- The most decisive in founding mathematics as we know it.

The impact of Greece is typified by the hyperbole of Sir Henry Main: “Except the blind forces of nature, nothing moves in this world which is not Greek in its origin.”² Including the adoption of Egyptian and other earlier cultures by the Greeks, we find their patrimony in all phases of modern life. Handicrafts, mining techniques, engineering, trade, governmental regulation of commerce and more have all come down to use from Rome and from Rome through Greece. Especially, our democracies and dictatorships go back to Greek exemplars, as well do our schools and universities, our sports, our games. And there is more. Our literature and literary genres, our alphabet, our music, our sculpture, and most particularly our mathematics all exist as facets of

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²Rede Lecture for 1875, in J.A. Symonds, *Studies of Greek Poets*, London, 1920.

the Greek heritage. The detailed study of Greek mathematics reveals much about modern mathematics, if not the modern directions, then the logic and methods.

The best estimate is that the Greek civilization dates back to 2800 BCE — just about the time of the construction of the great pyramids in Egypt. The Greeks settled in Asia Minor, possibly their original home, in the area of modern Greece, and in southern Italy, Sicily, Crete, Rhodes, Delos, and North Africa. About 775 BCE they changed from a hieroglyphic writing to the Phoenician alphabet. This allowed them to become more literate, or at least more facile in their ability to express conceptual thought. The ancient Greek civilization lasted until about 600 BCE. Originally, the Egyptian and Babylonian influence was greatest in Miletus, a city of Ionia in Asia Minor and the birthplace of Greek philosophy, mathematics and science.

From the viewpoint of its mathematics, it is best to distinguish between the two periods: the **classical** period from about 600 BCE to 300 BCE and the **Alexandrian** or **Hellenistic** period from 300 BCE to 300 A.D. Indeed, from about 350 BCE the center of mathematics moved from Athens to Alexandria (in Egypt), the city built by **Ptolemy I**, a Macedonian general in the army of **Alexander the Great** (358-323 BCE). It remained a center of mathematics for most of the next millennium, until the library was sacked by the Muslims in about 700 A.D.

1 The Sources of Greek Mathematics

In actual fact, our direct knowledge of Greek mathematics is less reliable than that of the older Egyptian and Babylonian mathematics, because none of the original manuscripts are extant.

There are two sources:

- Byzantine Greek codices (manuscript books) written 500-1500 years after the Greek works were composed.
- Arabic translations of Greek works and Latin translations of the Arabic versions. (Were there changes to the originals?)

Moreover, we do not know even if these works were made from the originals. For example, Heron made a number of changes in Euclid's *Elements*, adding new cases, providing different proofs and converses. Likewise for Theon of Alexandria (400 A. D.).

The Greeks wrote histories of Mathematics:

- **Eudemus** (4th century BCE), a member of Aristotle's school wrote histories³ of arithmetic, geometry and astronomy (lost),
- **Theophrastus** (c. 372 - c. 287 BCE) wrote a history of physics (lost).
- **Pappus** (late 3rd century CE) wrote the *Mathematical Collection*, an account of classical mathematics from Euclid to Ptolemy (extant).
- **Pappus** wrote *Treasury of Analysis*, a collection of the Greek works themselves (lost).
- **Proclus** (410-485 CE) wrote the *Commentary*, treating Book I of Euclid and contains quotations due to Eudemus (extant).
- various fragments of others.

2 The Major Schools of Greek Mathematics

The Classical Greek mathematics can be neatly divided in to several schools, which represent a philosophy and a style of mathematics. Culminating with *The Elements* of Euclid, each contributed in a real way important facets to that monumental work. In some cases the influence was much broader. We begin with the Ionian School.

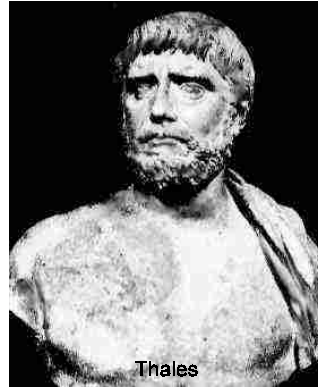
2.1 The Ionian School

The Ionian School was founded by Thales (c. 643 - c. 546 BCE). Students included Anaximander⁴ (c. 610 - c. 547 BCE) and

³Here the most remarkable fact must be that knowledge at that time must have been sufficiently broad and extensive to warrant histories.

⁴Anaximander further developed the air, water, fire theory as the original and primary form of the body, arguing that it was unnecessary to fix upon any one of them. He preferred

Anaximenes (c. 550 - c. 480 BCE), actually a student of Anaximander. He regarded air as the origin and used the term 'air' as god. Thales is the first of those to write on physics *physiologia*, which was on the principles of being and developing in things. His work was enthusiastically advanced by his student Anaximander. Exploring the origins of the universe, Anaximander wrote that the first principle was a vast Indefinite-Infinite (*apeiron*), a boundless mass possessing no specific qualities. By inherent forces, it gradually developed into the universe. In his system, the animate and eternal but impersonal Infinite is the only God, and is unvarying and everlasting.



Thales is sometimes credited with having given the first deductive proofs. He is credited with five basic theorems in plane geometry, one being that the every triangle inscribed in a semicircle is a right triangle. Another result, that the diameter bisects a circle appears in *The Elements* as a definition. Therefore, it is doubtful that proofs provided by Thales match the rigor of logic based on the principles set out by Aristotle and climaxed in *The Elements*. Thales is also credited with a number of remarkable achievements, from astronomy to mensuration to business acumen, that will be taken up another chapter.

The importance of the Ionian School for philosophy and the philosophy of science is without dispute.

2.2 The Pythagorean School

The Pythagorean School was founded by Pythagoras in about 455 BCE. More on this later. A brief list of Pythagorean contributions includes:

1. Philosophy.
2. The study of proportion.
3. The study of plane and solid geometry.

the *boundless* as the source and destiny of all things.

4. Number theory.
5. The theory of proof.
6. The discovery of incommensurables.⁵

For another example, Hippocrates of Chios (late 5th century BCE), computed the quadrature of certain lunes. This, by the way, is the first correct proof of the area of a curvilinear figure, next to the circle, though the issue is technical. He also was able to duplicate the cube by finding two mean proportionals. That is, take $a = 1$ and $b = 2$ in equation of two mean proportionals $a : x = x : y = y : b$. Solve for x to get $x = \sqrt[3]{2}$.)

2.3 The Eleatic School

The Eleatic School from the southern Italian city of Elea was founded by Xenophanes of Colophon, but its chief tenets appear first in Parmenides, the second leader of the school. Melissus was the third and last leader of the school. Zeno of Elea (c. 495 - c. 430 BCE), son of Teleutagoras and pupil and friend of Parmenides, no doubt strongly influenced the school. Called by Aristotle the inventor of dialectic, he is universally known for his four paradoxes. These, while perplexing generations of thinkers, contributed substantially to the development of logical and mathematical rigor. They were regarded as insoluble until the development of precise concepts of continuity and infinity.

It remains controversial that Zeno was arguing against the Pythagoreans who believed in a plurality composed of numbers that were thought of as extended units. The fact is that the logical problems which his paradoxes raise about a mathematical continuum are serious, fundamental, and were inadequately solved by Aristotle.

Zeno made use of three premises:

1. Any unit has magnitude
2. That it is infinitely divisible
3. That it is indivisible.

⁵The discovery of incommensurables brought to a fore one of the principle difficulties in all of mathematics — the nature of infinity.

Yet he incorporated arguments for each. In his hands, he had a very powerful complex argument in the form of a dilemma, one horn of which supposed indivisibility, the other infinite divisibility, both leading to a contradiction of the original hypothesis who brought to the fore the contradictions between the discrete and the continuous, the decomposable and indecomposable.

Zeno's Paradoxes

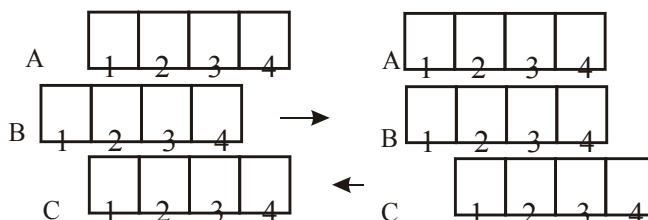
Zeno constructed his paradoxes to illustrate that current notions of motion are unclear, that whether one viewed time or space as continuous or discrete, there are contradictions. Paradoxes such as these arose because mankind was attempting to rationally understand the notions of infinity for the first time. The confusion centers around what happens when the logic of the finite (discrete) is used to treat the infinite (infinitesimal) and conversely, when the infinite is perceived within the discrete logical framework. They are

Dichotomy. To get to a fixed point one must cover the halfway mark, and then the halfway mark of what remains, etc.

Achilles. Essentially the same for a moving point.

Arrow. An object in flight occupies a space equal to itself but that which occupies a space equal to itself is not in motion.

Stade. Suppose there is a smallest instant of time. Then time must be further divisible!



Stade

Now, the idea is this: if there is a smallest instant of time and if the farthest that a block can move in that instant is the length of one block, then if we move the set B to the right that length in the smallest instant and the set C to the left in that instant, then the net shift of the sets B and C is two blocks. Thus there must be a smaller instant

of time when the relative shift is just one block. Note here the use of relative motion.

Aristotle gave these paradoxes considerable time, in his effort to resolve them. His resolutions are unsatisfactory from a modern viewpoint. However, even today the paradoxes puzzle and confound anyone uninitiated in the ways of “limits, continuity, and infinity.”

Democritus of Abdera (ca. 460-370 BCE) should also be included with the Eleatics. One of a half a dozen great figures of this era, he was renown for many different abilities. For example, he was a proponent of the materialistic atomic doctrine. He wrote books on numbers, geometry, tangencies, and irrationals. (His work in geometry was said to be significant.) To demonstrate the clarity of thinking at the time, both he and Protagoras puzzled over whether the tangent to a circle meets it at a point or a line. By the time of Euclid, this subtle point was eventually settled in favor of “a point”, but it is effectively a definition — not a proposition. In addition, he discovered⁶ that the volumes of a cone and a pyramid are $1/3$ the volumes of the respective cylinder and prism.

2.4 The Sophist School

The Sophist School (~480 BCE) was centered in Athens, just after the final defeat of the Persians.⁷ There were many sophists and for many years, say until 380 BCE, they were the only source of higher education in the more advanced Greek cities. Of course such services were provided for money. Their influence waned as the philosophic schools, such as Plato’s academy, grew in prestige. Chief among the sophists were most important were Protagoras, Gorgias, Antiphon, Prodicus, and Thrasymachus. In some regards Socrates must be considered among them, or at least a special category of one among them. Plato emphasized, however, that Socrates never accepted money for knowledge.

Greece

⁶as attested by Archimedes. However, he did not rigorously prove these results. Recall that the formula for the volume pyramid was known to the Egyptians and the Babylonians.

⁷This was the time of Pericles. Athens became a rich trading center with a true democratic tradition. All citizens met annually to discuss the current affairs of state and to vote for leaders. Ionians and Pythagoreans were attracted to Athens. This was also the time of the conquest of Athens by Sparta.

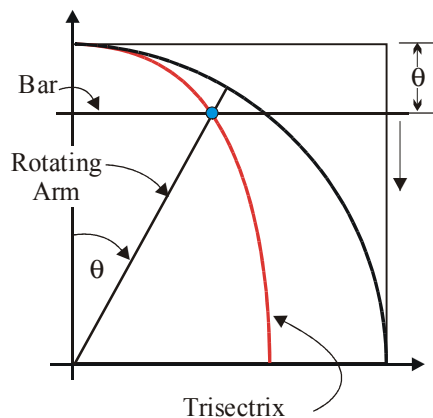
Because Athens was a democracy, young men needed instruction in politics. Sophists provided that instruction, teaching men how to speak and what arguments to use in public debate. A Sophistic education became popular among older families and the upwardly mobile without families. Among the instruction given were ways to argue against traditional values, which Plato thought unfair and unjustified. However, he learned that even to defend traditional values, one must use a reasoned argument, not appeals to tradition and unreflecting faith.

In the Sophist school, emphasis was given to abstract reasoning and to the goal of using reason to understand the universe. This school had amongst its chief pursuits the use of mathematics to understand the function of the universe. At this time many efforts were made to solve the three great problems of antiquity: doubling the cube, squaring the circle, and trisecting an angle — with just a straight edge and compass. One member of this school who ventured a solution to the angle trisection problem, was the mathematician Hippias of Elis (460 - 400 BCE).⁸ For example, the Sophists Antiphon of Rhamnos (c. 480 - 411 BCE) and Bryson of Heraclea (b. 450?) considered the circle squaring problem by comparing the circle to polygons inscribed within it. Another sophist, Hippias of Elis lectured widely on mathematics and as well on poetry, grammar, history, politics, archaeology, and astronomy. He was a prolific writer, producing elegies, tragedies, and technical treatises in prose. His work on Homer was considered excellent. Nothing of his remains except a few fragments.

According to Plato, Hippias whom he depicted in his dialogue *Protagoras*, was a 'vain and boastful man' (ca. 460 BCE) who discovered the trisectrix. The trisectrix, also known as the quadratrix, was a mechanically generated curve which he showed could be used to trisect any angle.

The Trisectrix. Here is how to construct the trisectrix. A rotating arm begins at the vertical position and rotates clockwise as a constant rate to the 3 o'clock position. A horizontal bar falls from the top (12 o'clock position) to the x -axis at a constant rate, in the same time. The locus of points where the horizontal bar intersects the rotating arm traces the trisectrix. (In the figure below, you may assume that radian measure is used with a (quarter) circle of radius $\frac{\pi}{2}$. Thus the time axis ranges in $[0, \frac{\pi}{2}]$.)

⁸a city in the Peloponnesus



2.5 The Platonic School

The Platonic School, the most famous of all was founded by **Plato** (427 - 347 BCE) in 387 BCE in Athens as an institute for the pursuit of philosophical and scientific teaching and research. Plato, though not a mathematician, encouraged research in mathematics. Pythagorean fore-runners of the school, **Theodorus⁹ of Cyrene** and **Archytas¹⁰ of Tarentum**,

⁹Theodorus proved the incommensurability of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, ..., $\sqrt{17}$.

¹⁰Archytas solved the duplication of the cube problem at the intersection of a cone, a torus, and a cylinder.

through their teachings, produced a strong Pythagorean influence in the entire Platonic school. Little is known of Plato's personality and little can be inferred from his writing. Said Aristotle, certainly his most able and famous student, Plato is a man "whom it is blasphemy in the base even to praise." This meant that even those of base standing in society should not mention his name, so noble was he. Much of the most significant mathematical work of the 4th century was accomplished by colleagues or pupils of Plato.



Members of the school included **Menaechmus**¹¹ and his brother **Dinostratus**¹² and **Theaetetus**¹³(c. 415-369 BCE) According to Proclus, Menaechmus was one of those who "made the whole of geometry more perfect". We know little of the details. He is also attributed the quote to Alexander

"O King, for traveling over the country there are royal roads and roads for common citizens; but in geometry these is one road for all."

This famous quotation, in slightly different forms, has been attributed to others, as well, notably Euclid. As the inventor of the conics Menaechmus no doubt was aware of many of the now familiar properties of conics, including asymptotes. He was also probably aware of the solution of the duplication of the cube problem by intersecting the parabola $y^2 = 2ax$ and the hyperbola $xy = a^2$, for which the solution is $x = a\sqrt[3]{2}$. For, solving both for x yields

$$\begin{aligned} x &= y^2/(2a) \quad \text{and} \quad x = a^2/y \\ y^2/(2a) &= a^2/y \end{aligned}$$

¹¹Menaechmus invented the conic sections. Only one branch of the hyperbola was recognized at this time.

¹²Dinostratus showed how to square the circle using the trisectrix

¹³Theaetetus proved that there are only five regular solids: the tetrahedron (4 sides, triangles), cube (6 sides, squares), octahedron (8 sides, triangles), dodecahedron (12 sides, pentagons), and icosahedron (20 sides, hexagons). Theaetetus was a student of Theodorus

$$\begin{aligned}y^3 &= 2a^3 \\y &= \sqrt[3]{2} \cdot a.\end{aligned}$$

Another famous pupil/friend, **Eudoxus** of Cnidos removed his school from Cyzicus to Athens for the purpose of cooperating with Plato. During one of Plato's absences Eudoxus apparently acted as the head of the Academy.

The academy of Plato was much like a modern university. There were grounds, buildings, students, and formal course taught by Plato and his aides. During the classical period, mathematics and philosophy were favored. Plato was not a mathematician — but was a strong advocate of all of mathematics. Plato believed that the **perfect ideals** of physical objects are the reality. The world of ideals and relationships among them is permanent, ageless, incorruptible, and universal. The Platonists are credited with discovery of two methods of proof, the method of analysis¹⁴ and the *reductio ad absurdum*.¹⁵

Plato affirmed the deductive organization of knowledge, and was first to systematize the rules of rigorous demonstration.

The academy was closed by the Christian emperor Justinian in A.D. 529 because it taught “pagan and perverse learning”.

2.6 The School of Eudoxus

The School of Eudoxus founded by Eudoxus (c. 408 BCE), the most famous of all the classical Greek mathematicians and second only to Archimedes.

- Eudoxus developed the theory of proportion, partly to account for and study the *incommensurables* (irrationals).
- He produced many theorems in plane geometry and furthered the logical organization of proof.
- He also introduced the notion of *magnitude*.

¹⁴where what is to be proved is regarded as known and the consequences deduced until a known truth or a contradiction is reached. A contradiction renders the proposition to be false.

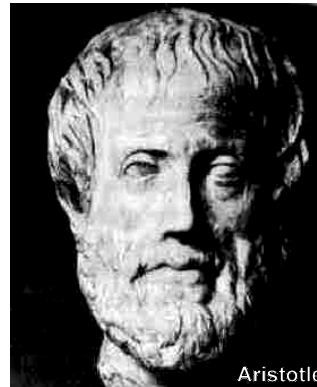
¹⁵where what is to be proved is taken and false and consequences are deduced until a contradiction is produced, thus proving the proposition. This, the indirect method, is also attributed to Hippocrates.

- He gave the first rigorous proof on the quadrature of the circle. (Proposition. The areas of two circles are as the squares of their diameters. ¹⁶)

2.7 The School of Aristotle

The School of Aristotle, called the Lyceum, founded by Aristotle (384-322 BCE) followed the Platonic school. It had a garden, a lecture room, and an altar to the Muses.

Of his character more is known than for others we have considered. He seems to have been wealthy with holdings from Stagira. Therefore, he had the leisure to study. He apparently used sums of money to purchase books. So many books did he read that Plato referred to him as the “reader”, indicating a bit of contempt or perhaps rivalry. While still a member of Plato’s academy,



his early writings were dialogues concerned with thoughts of the next world and the worthlessness of this one, themes familiar to him from Plato’s writing (e.g. *Phaedo*). Anecdotes about him show him as a kindly and affectionate. They show hardly a trace of the self-importance that some scholars claim to detect in his works. His will has survived and exhibits the same kindly traits; he references a happy family life and takes solicitous care of his children, as well as his servants. His apparent joy of life is reflected in the literary *On Philosophy*, which was completed in about 348. Afterwards, he devoted his energies to research, teaching, and writing of technical treatises.

After Plato’s death in about 348, his (Plato’s) nephew Speusippus was appointed head of the Academy. Shortly thereafter Aristotle left Athens, possibly as some claim because of not being appointed Plato’s successor. He travelled, with friend Xenocrates to Assus where he enjoyed the patronship of Hermeias of Atarneus, a Greek soldier of fortune. There he was a principal at the new Assus Academy. Here he wrote much including *On Politics* and *On Kinship* (now lost). After just three

¹⁶At this time there is still no apparent concept of a formula such as $A = \pi r^2$.

years at the Assus Academy, Aristotle moved to the island of Lesbos and settled in Mytilene, the capital city. With his friend Theophrastus he established a philosophical circle similar to the Athenian Academy. In late 343 at the age of 42, Aristotle was invited by Philip II of Macedonia to his capital at Pella to tutor his 13-year-old son, Alexander. As the leading intellectual figure of the day, Aristotle was instructed to prepare Alexander for his future role as a military leader and king.

After three years in Pella, Aristotle returned to Stagira and remained there until 335, when at almost 50 years of age, he returned to Athens. There he opened his own academy, the Lyceum, a gymnasium attached to the temple of Apollo Lyceus. It was situated in a grove just outside Athens. It was a place frequented by teachers, including possibly Socrates. It was only after Aristotle's death that the school, under Theophrastus, acquired extensive property. His instruction, given in the peripatos, or covered walkway, of the gymnasium, was the source of the namesake of Peripatetic.

In 323, with the death of Alexander, there was some anti-Macedonian sentiment throughout Athens. Well connected to the Macedonians through Alexander, Aristotle fled the city to his mother's estates in Chalcis on the island of Euboea where within a year at the age of 62 or 63 he died from a stomach illness. Referring obviously to Socrates, it was reported that he left Athens in order to save the Athenians from sinning twice against philosophy.

Aristotle's writings fall into two groups. The first group is comprised of those works published by Aristotle and now lost. The second group consists of those not published nor intended for publication by Aristotle but collected and preserved by others. Finally, the writings that have survived, termed "acroamatic," or treatises, were intended for use in Aristotle's school and were written in a concise and individualistic style. In later antiquity Aristotle's collected writings totalled hundreds of rolls. Today the surviving 30 works fill more than 2,000 printed pages. Ancient catalogs list more than 170 separate works by Aristotle.

Aristotle set the philosophy of physics, mathematics, and reality on a foundations that would carry it to modern times. He viewed the sciences as being of three types — **theoretical** (math physics, logic and metaphysics), **productive** (the arts), and the **practical** (ethics, politics). He contributed little to mathematics however,

...his views on the nature of mathematics and its relations to the physical world were highly influential. Whereas Plato believed that there was an independent, eternally existing world of ideas which constituted the reality of the universe and that mathematical concepts were part of this world,¹⁷ Aristotle favored concrete matter or substance¹⁸.

Aristotle regards the notion of **definition**¹⁹ as a significant aspect of argument. He required that a definition may not reference prior objects. The following definition,

'A point is that which has no part',

which is the first definition from the first book of Euclid's *Elements*²⁰, would be unacceptable. Aristotle also treats the basic principles of mathematics, distinguishing between **axioms** and **postulates**.

- Axioms include the laws of logic, the law of contradiction, etc.
- The postulates need not be self-evident, but their truth must be sustained by the results derived from them.

Euclid uses this distinction. Aristotle explored the relation of the point to the line — again the problem of the indecomposable and decomposable.

Aristotle makes the distinction between **potential infinity** and **actual infinity**²¹. He states only the former actually exists, in all regards.

Aristotle is credited with the invention of logic, through the syllogism. He cites two laws studied by every student.

1. The law of contradiction. (A statement may not be T and F)
2. The law of the excluded middle. (A statement must be T or F, there is no other alternative.)

¹⁷This is still an issue of debate and contention

¹⁸Morris Kline, *Mathematical Thought From Ancient to Modern Times*

¹⁹and hence also undefined terms

²⁰This self-referential use of language in definition and statement is the logician's bane of language, having caused paradoxes and other problems well into the 20th century

²¹This is still a problem today.

His logic remained unchallenged until the 19th century. Even Aristotle regarded logic as an independent subject that should precede science and mathematics.

Aristotle's influence has been immeasurably vast.

3 Exercises

1. Show how to trisect a line segment.
2. Show how to trisect an angle using the trisectrix. (Hint. First trisect the *line* segment projected to the right margin by the angle to the angle that is to be trisected.)