Number Theory Section Summary: 1.1-3

Some Preliminary Considerations

"The theory of numbers is concerned, at least in its elementary aspects, with properties of the integers and more particularly with the positive integers 1, 2, 3, ...(also known as the *natural numbers*)...." p. 1.

1. Definitions

Well-Ordering Principle: Every nonempty set S of non-negative integers contains a least element; that is, there is some integer a in S such that $a \leq b$ for every b belonging to S.

2. Theorems

Archimedean property: If a and b are any positive integers, then there exists a positive integer n such that $na \geq b$.

(Proof is by **contradiction**).

Suppose there are two integers a + b that don't have this property.

Consider $S = \{ b - na \mid n \in \mathbb{N} \}$, a non-empty subset of $\mathbb{N}$. So S has a least element, $b - ma$, by well-ordering.

Consider $b-(m+1)a = b - ma - a < b - ma$; but $b-(m+1)a \in S$, so this contradicts the existence of the least element; if S were non-empty it would have to have one.

This contradiction means that there is no such pair a + b; so the Archimedean property holds.

First Principle of Finite Induction: Let S be a set of positive integers with the properties that

(a) The integer 1 belongs to S, and

(b) whenever the integer k is in S, then the next integer $k+1$ is also in S.

Then S is the set of all positive integers.

Our text assumes the Well-Ordering Principle, and uses it to prove mathematical induction; alternatively we could prove the Well-Ordering Principle based on assuming Mathematical Induction. This means that the two are **equivalent**: We have Well-Ordering $\iff$ Induction.

Assume the ~~1st~~ 2nd Principle of Finite induction.

Now consider A, a nonempty subset of $\mathbb{N}$.

By the approach of contradiction, assume that A has no least element. So $1 \notin A$, or it would be the least element.

Consider the set B of all natural numbers not in A. Suppose that $k \in B$. $k+1 > k$. Furthermore, suppose that every natural number up to k is in B. Then $k+1 \notin A$, or it would be the least element! But A has no least element! So $k+1 \in B$. By the 2nd principle of finite induction, every natural number is in B.

A is empty, a contradiction.

A must have a least element, which establishes well-ordering.

Second Principle of Finite Induction: Let S be a set of positive integers with the properties that

(a) The integer 1 belongs to S, and

(b) whenever the integers $1, \ldots, k$ are in S, then the next integer $k+1$ is also in S.

Then S is the set of all positive integers.

(equivalent to the 1st principle, although it seems more restrictive).

Binomial Theorem:

$$\overline{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(Proof is by **induction**).

3. Properties/Tricks/Hints/Etc. **Pascal's rule**:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

(this is the source of **Pascal's triangle**).

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1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

4. Summary

Chapter 1 is preliminary, as the title says. We assume that you've seen this stuff before (except for the history of number theory, which I hope that you'll find interesting!).

The story concerning the drowned disciple of Pythagoras is often told of another student, who may have revealed the existence of irrational numbers (in particular, $\sqrt{2}$). Irrational numbers were not welcomed into polite Pythagorean society...!

1a) p6 Gauss

Prove $1+2+\dots+n = \frac{n(n+1)}{2}$, $n \geq 1$

By induction:

Anchor: $n=1$

$$1 = \frac{1(1+1)}{2} \checkmark$$

Induction:

Suppose the result holds for $n=k$.

Show that it holds for $k+1$.

$$1+\dots+k = \frac{k(k+1)}{2}$$

Consider the $k+1^{th}$ case:

$$\underbrace{1+\dots+k}_{\frac{k(k+1)}{2}} + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2}$$

So it works for $n=k+1$, & hence for all natural numbers.

#3a p 11

$$\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n} = 2^n$$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$(1+1)^n = 2^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}\underbrace{1^{n-k}1^k}_{1} = \sum_{k=0}^{n}\binom{n}{k}$$

3b $\displaystyle\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$ use the strategy of 3a.

1 a + c p 15

1a) x triangular $\iff$ $x = \dfrac{n(n+1)}{2}$ for some $n \geq 1$

triangular $\iff$ $x = \displaystyle\sum_{k=1}^{n} k = \underbrace{\frac{n(n+1)}{2}}_{\text{Gauss}}$

1c) Sum of consecutive triangular numbers is a perfect square.

Consider $t_k = \frac{k(k+1)}{2}$

$$t_{k+1} = \frac{(k+1)((k+1)+1)}{2}$$

$$t_k + t_{k+1} = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(2k+2)}{2} = (k+1)^2$$