Number Theory Section Summary: 6.1
Number-Theoretic Functions

1. Summary

We encounter two interesting number-theoretic functions, τ and σ, and discover an interesting relationship between these and the prime factorization of a number.

The concept of a multiplicative function is also introduced, which will prove useful (now and later on).

2. **Definitions**

τ - tau
σ - sigma

Number-theoretic function: any function whose domain is the set of natural numbers (and whose range is generally also in $\mathbb{N}$).

Definition 6.1: Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n, and $\sigma(n)$ denote the sum of those divisors.

The notation

$$\sum_{d|n} f(d)$$

means "sum the values of f as d runs over the divisors of n". Given that, then

$$\tau(n) = \sum_{d|n} 1$$

– counting the divisors d of n

and

$$\sigma(n) = \sum_{d|n} d$$

adds up the divisors

Example: : Evaluate $\tau(24)$ and $\sigma(24)$.

Divisors:

1, 2, 3, 4, 6, 8, 12, 24

$\tau(24) = 8$

$\sigma(24) = 60$

Example: : Evaluate $\tau(240)$ and $\sigma(240)$. $\boxed{240 = 2^4 \cdot 3 \cdot 5}$

Divisors : 1, 2, 3, 4, 5, 6, 8, 10, 12, 24, 15, 16

20, 30, 40, 48, 60, 80, 120, 240

$\tau(240) = 20$

$\sigma(240) = 744$

Example: : What are $\tau(p)$ and $\sigma(p)$ when p is prime?

$\tau(p) = 2$

$\sigma(p) = p+1$

Example: #15, p. 110

$n,\ n+2$ twin primes

$$\sigma(n+2) = (n+2)+1 \qquad \sigma(n) = n+1$$

$$= (n+1)+2$$

$$= \sigma(n)+2$$

Also holds for $n = 434$ & 8575

Example: : How do $\tau(4)\tau(6)$ and $\tau(24)$ compare?

$= 8$

$\tau(4) = 3$ $\quad$ $\tau(6) = 4$

$\tau(4)\cdot\tau(6) = 12 \neq \tau(24)$

$\tau(21) = 4 = \tau(3)\cdot\tau(7)$

$\tau(3) = 2 \quad \tau(7) = 2$

$\tau(24) = 8$

$\tau(3) = 2$

$\tau(8) = 4$

Definition 6.2: A number-theoretic function is said to be **multiplicative** if

$$f(mn) = f(m)f(n)$$

whenever $\gcd(m, n) = 1$.

Examples: $f(n) = 1$, $g(n) = n$.

$f(mn) = 1 = 1\cdot 1$

$= f(m)\cdot f(n)$ ✓

$g(mn) = mn = g(m)\cdot g(n)$ ✓

By induction,

$$f(n_1 n_2 \cdots n_r) = f(n_1)f(n_2)\cdots f(n_r)$$

whenever the n_i are pairwise relatively prime. Hence, a multiplicative function is completely determined for n once its values on the prime powers of the factorization of n are known:

$$f(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = f(p_1^{k_1})f(p_2^{k_2})\cdots f(p_r^{k_r})$$

This is why it's a useful notion

Example: #17, p. 110

$f(x) = x^k$ is multiplicative

$f(mn) = (mn)^k = m^k n^k = f(m) \cdot f(n)$

$f(x) = 1 \Rightarrow k = 0$

$f(x) = x \Rightarrow k = 1$

3. Theorems

Theorem 6.1 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then the positive divisors of n are precisely those integers of the form $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $0 \le a_i \le k_i$ for i in $\{1, \ldots, r\}$.

Theorem 6.2 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a)

$$\boxed{\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)}$$

and

(b)

$$\boxed{\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdots \frac{p_r^{k_r+1} - 1}{p_r - 1}}$$

The proof of the first is a counting argument, and the second uses a sum of a geometric series and a neat decomposition.

$$\sigma(n) = \underbrace{(1 + p_1 + \cdots + p_1^{k_1})}\underbrace{(1 + p_2 + \cdots + p_2^{k_2})} \cdots \underbrace{(1 + p_r + \cdots + p_r^{k_r})}$$

$$= \frac{p_1^{k_1+1} - 1}{p_1 - 1} \quad \frac{p_2^{k_2+1} - 1}{p_2 - 1} \quad \cdots \quad \frac{p_r^{k_r+1} - 1}{p_r - 1}$$

$$\sigma(240) = \frac{2^5-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{5^2-1}{5-1} = 744$$

$$31 \cdot 4 \cdot 6$$

The notation

$$\prod_{i=1}^{r} f(i)$$

$$\boxed{n! = \prod_{i=1}^{n} i}$$

means "multiply the values of f as i runs over from 1 to r". Given that, then

$$\tau(n) = \prod_{i=1}^{r}(k_i+1)$$

and

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{k_i+1}-1}{p_i-1}$$

Let's check for $n = 240$. ✓

Theorem 6.3 The functions τ and σ are multiplicative functions.

Lemma If $\gcd(m,n) = 1$, then the set of positive divisors of mn consists of all products $d_1 d_2$, where $d_1|m$, $d_2|n$, and $\gcd(d_1,d_2) = 1$; furthermore these products are all distinct.

Theorem 6.4 If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d|n} f(d)$$

then F is also multiplicative.

Corollary: the functions τ and σ are multiplicative functions.

$$\tau(n) = \sum_{d|n} 1$$

multiplicative function $f(d)=1$

$$\sigma(n) = \sum_{d|n} d$$

identity function $f(d) = d$ is multiplicative

Demonstration of Thm 6.4.

Given f multiplicative & $F(n) = \sum\limits_{d \mid n} f(d)$, & m & n relatively prime.

$$\underbrace{(d_1 + \dots + d_r)}_{\text{divisors of } m}\underbrace{(\delta_1 + \dots + \delta_s)}_{\text{divisors of } n} = (\text{sum of}) \text{ all the divisors of } mn$$

distinct

$$F(mn) = \sum_{d \mid mn} f(d)$$

$$= \sum_{i \mid m} \sum_{j \mid n} f(i \cdot j)$$

$$= \sum_{i \mid m} \sum_{j \mid n} f(i) \cdot f(j) \qquad [\gcd(i,j) = 1]$$

$$= \sum_{i \mid m} f(i) \cdot \sum_{j \mid n} f(j)$$

$$= F(m) \cdot F(n)$$