The Curl of a Vector Field. Stokes' Theorem expresses the integral of a vector field **F** around a closed curve as a surface integral of another vector field, called the **curl** of **F**. This vector field is constructed in the proof of the theorem. Once we have it, we invent the notation $\nabla \times \mathbf{F}$ in order to remember how to compute it. In this notation ∇ stands for the **vector operator**

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

and expressions containing ∇ are interpreted by first pretending that it is an ordinary vector, so that vector operations will introduce terms that abut a component from the first factor with one from the second factor. For numerical vectors, this is interpreted as multiplication, but for ∇ the interpretation is to let the component of ∇ operate on the component from the other factor.

The **gradient** also uses this interpretation of ∇ to construct a vector field from a scalar function. The **curl** takes one vector field to another using a construction modeled by the cross product.

A related operation, called the **divergence** will be introduced later.

Each of these acts as a derivative, and there is a version of the fundamental theorem that evaluates an integral of this derivative.

The **fundamental theorem for line integrals** says that the difference of a scalar function $f(x, y, z)$ at two points is the integral of ∇f (the gradient of f) along **any** path joining the two points. From this it follows that the integral of ∇f is **independent of path** and that the integral of ∇f along any closed path is zero.

Stokes' theorem says that the integral of a vector field $\mathbf{F}(x, y, z)$ along a closed path is equal to the integral of $\nabla \times \mathbf{F}$ (the curl of **F**) along **any** surface having the given path as its **positively oriented** boundary. From this it follows that the integral of $\nabla \times \mathbf{F}$ is zero on any closed surface.

The **divergence theorem** (to be discussed later) relates the integral of a vector field over a closed surface to the integral of its divergence over the three dimensional region bounded by the surface. Although the details are different, all three theorems have similar statements.

Special second derivatives. It has already been noted that

$$\nabla \times (\nabla f) = 0.$$

Formally, this is just the equality of mixed partials, but it is tied to Stokes' Theorem. If $\mathbf{F} = \nabla f$, the line integral of **F** along any curve is the difference of the values of f at the endpoints. For a closed curve, this is always zero. Stokes' Theorem then says that the surface integral of its curl is zero for every surface, so it is not surprising that the curl itself is zero.

Stokes' theorem also says that the integral of the curl of a vector field over a closed surface is zero. If we try to write a given vector field G as the curl of another vector field F, we will meet an **obstruction** to completing the computation in the form of a combination of derivatives of components of G that must be zero.

Exercises 16.5. Find curl of the following vector fields.

#1. $\langle xy, yz, zx \rangle$.

#5. $\langle e^x \sin y, e^x \cos y, z \rangle$.

#7. $\left\langle \frac{x}{z}, \frac{y}{z}, \frac{1}{z} \right\rangle$.

For any of these whose curl is zero, express as a gradient.

Finding a vector field from its curl. The first four exercises of Section 16.8 have the form: use Stokes' Theorem to evaluate

$$\iint_S \nabla \times \mathbf{F}\, d\mathbf{S}.$$

In this form, there isn't much to the exercise. This way of stating the exercise gives **F**, so its curl need never be computed since you must evaluate the line integral in order to feel that you have **used** Stokes' Theorem. There is still something to be done: you need to produce a parameterization of the positively oriented boundary from a description of the surface, but the statement of the exercise obscures its real content.

It would be more interesting to start from a vector field **G** that is of the form $\mathbf{G} = \nabla \times \mathbf{F}$, and determine **F**. Although this is more complicated than the process for writing a conservative vector field F as $F = \nabla f$, it has a similar flavor.

We now describe a solution of the equation

$$\nabla \times \mathbf{F} = \mathbf{G}$$

for **F** when **G** is given.

It is useful to simplify the problem as much as possible before beginning. We know that **F** is only defined up to a term of the form ∇f, and there is no difficulty (in principle, although it does require integration) finding a function f for which $f_3(x, y, z)$ is any expression. In particular, any solution **F** could be replaced by $\mathbf{F} - \nabla f$, where $f_3(x, y, z)$ is equal to the third component of **F**. This means that we need only look for

$$\mathbf{F} = \langle X, Y, 0\rangle.$$

Let

$$\mathbf{G} = \langle P, Q, R\rangle.$$

Looking at the components of the curl, we have

$$\begin{aligned} -Y_z &= P \\ X_z &= Q \\ Y_x - X_y &= R. \end{aligned} \qquad (*)$$

If the first of the equations in $(*)$ is solved for Y (up to a function of x and y), and this result put into the third equation, we are left with the task of finding X from its derivatives with respect to y and z. This was exactly what we did in recognizing conservative vector fields as gradients. The only requirement is that the equations $X_{zy} = X_{yz}$ must hold. These equations are

$$\begin{aligned} X_{zy} &= Q_y \\ X_{yz} &= Y_{xz} - R_z. \end{aligned}$$

Finally,

$$Y_{xz} = Y_{zx} = -P_x,$$

so

$$Q_y = -P_x - R_z.$$

Note that this condition depends only on **G** and not on the particular choice of Y.

If this is satisfied, then we can find X for any Y, which completes the determination of F.

This shows that $\mathbf{G} = \langle P, Q, R\rangle$ satisfies $G = \nabla \times \mathbf{F}$ if and only if $P_x + Q_y + R_z = 0$. This looks like it should be denoted $\nabla \cdot \mathbf{G}$.

The whole story is a little more subtle. It is easy to verify that $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, but the constructive part of these equivalences assumed that the functions obtained in the solution were **defined everywhere**. For objects defined only in part of space, the necessary condition my hold but it may not be possible to perform the construction.

An example. From exercise 1 in Section 16.5, we know that$\nabla \cdot \mathbf{G} = 0$ when $\mathbf{G} = \langle yz, xz, xy\rangle$. The first equation of $(*)$ becomes $Y_z = -yz$, so we take $Y = -yz^2/2$. This leaves

$$\begin{aligned} X_z &= xz \\ X_y &= -xy \end{aligned}$$

(since $Y_x = 0$). Then $X = xz^2/2 + f(x, y)$ and $f_2(x, y) = -xy$. The solution can be completed because the expression that is supposed to be $f_2(x, y)$ really is independent of z. Integrating this, gives $f(x, y) = -xy^2/2$ as one solution. Thus,

$$\mathbf{F} = \frac{1}{2}\Big\langle xz^2 - xy^2, -yz^2, 0\Big\rangle.$$

If we subtract $\nabla(x^2z^2/4) = \Big\langle xz^2/2, 0, x^2z/2\Big\rangle$ from this, we have the more symmetric solution

$$\mathbf{F} = -\frac{1}{2}\Big\langle xy^2, yz^2, zx^2\Big\rangle.$$

Additional Exercises. Write the given vector field **G** as $\mathbf{G} = \nabla \times \mathbf{F}$, if possible.

(A) $\mathbf{G} = \langle 0, 0, 1\rangle$.

(B) $\mathbf{G} = \Big\langle x^2, y^2, 0\Big\rangle$.

(C) $\mathbf{G} = \langle 0, 0, xe^y\rangle$.

Why is it called "curl"? Consider the vector field

$$\mathbf{R} = \langle -y, x, 0\rangle$$

that points in the positive direction along each circle with center at the origin, with a magnitude proportional to the radius. Such a motion corresponds to our image of *pure rotation*. An easy computation shows that

$$\nabla \times \mathbf{R} = \langle 0, 0, 2 \rangle,$$

identifying the axis of rotation everywhere. A physical model uses a small paddle-wheel moving in the flow $\mathbf{R}$. When the axis of the wheel is lined up with the z-axis, the difference in speeds at the ends of the paddle causes to paddle spin relative to coordinates with fixed directions as its center of mass moves with the flow. This behavior is seen throughout the flow — not just near the center of rotation.

The rotation is detected in the vector field $\mathbf{R}$ itself by forming the line integral of $\mathbf{R}$ along closed curves. Stokes' theorem tells us that such an integral is given by the integral of $\nabla \times \mathbf{R}$ over a surface bounded by the curve. In particular, the value of the integral may be estimated by the product of the area of the surface and the component of $\nabla \times \mathbf{R}$ perpendicular to the surface. For a small circle of radius r, the area is proportional

to r^2, so this effect is not overwhelming. However when considered on a fixed scale, the magnitude of the curl gives a noticeable rotation. To be free of such effects, a vector field must have zero curl.