MAT320 Final, Spring 2004:
Featuring Chapter 17, plus the old stuff...

Name:

Directions:

- You may use a 3x5 (inches, please!) card: please include it with your test.

- All problems are equally weighted. The test is in two sections:

    - you must skip **two** of the first eight problems (write "Skip" plainly on those you wish to skip). If you do NOT write "skip" on two problems, then you will suffer untold but extremely painful karmic consequences.

    - You must answer all six of the remaining problems (numbered nine to fourteen).

- Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning).

- If a problem has been reduced to univariate integrals (that aren't trivially evaluated, in which case you **should** evaluate them), then you may use your calculator to evaluate them. This exam is not about integration by parts!

- Indicate clearly your answer to each problem (e.g., put a box around it).

- **Good luck!**

Problem 1 Consider the function
$$g(x, y) = xy^2 - e^y$$

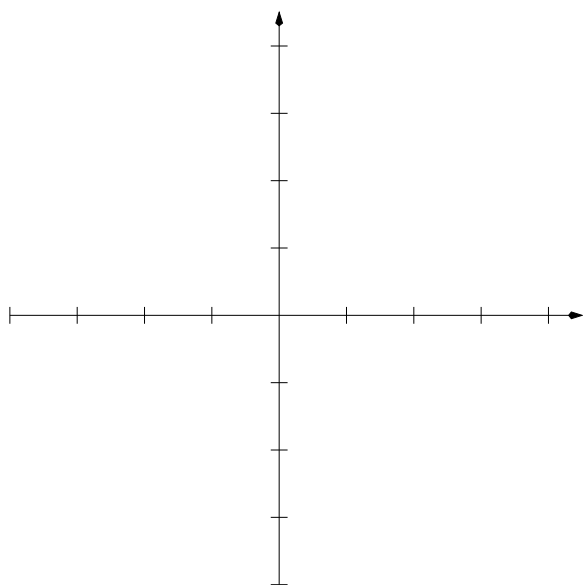- Compute the gradient of g, and evaluate it at the origin.

- Compare the mixed partials g_{xy} and g_{yx} – should they be the same (why or why not)?

- What is the directional derivative of g in the direction of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$?

Problem 2 Draw (and label!) several contours of the function
$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

Does the limit of f as $(x, y) \to 0$ exist? Demonstrate!

Problem 3 Compute the integral of $f(x, y) = \sin(x^2 + y^2)$ over the circle of radius $\sqrt{\frac{\pi}{2}}$ centered at the origin.

Problem 4 Find the center of mass of the bottom half (negative z values) of the unit sphere centered at the origin. (Use symmetry!)

Problem 5 Write the iterated integral

$$I = \int_0^2 \int_{\frac{x}{2}-1}^{3-\frac{3x}{2}} dy\, dx$$

as a sum of two iterated integrals in which the first integration is with respect to x. How can we compute the value of I without formally integrating? (Draw the region!)
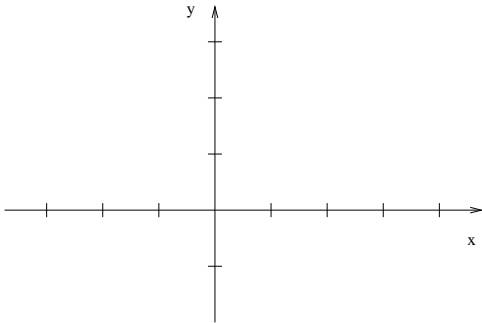
Problem 6 Find and classify the extrema of the function $f(x, y, z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.

Problem 7 Find and classify the extrema of the function $f(x, y) = xy(1 - x - y)$.

Problem 8 Describe and draw the object whose volume is given by the integral

$$I = \int_0^\pi \int_0^{\frac{\pi}{6}} \int_0^2 dV,$$

where dV is the volume element in spherical coordinates (ending in $d\rho d\phi d\theta$); then evaluate the integral.

Problem 9 Consider the vector function $\mathbf{F} = \langle y \ln(z) - z, x \ln(z) + 1, \frac{xy}{z} - x \rangle$.

- Find the domain of $\mathbf{F}$, and demonstrate that $\mathbf{F}$ is conservative on its domain.

- What is $\nabla \times \mathbf{F}$?

- What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle parameterized by $\mathbf{r}(t) = \langle 2 + \cos(t), 2 + \sin(t), 2 \rangle$? (What if we had centered the circle at $(2, 2, 0)$?

Problem 10 A particle starts at the point (-2,0), moves along the x-axis to (2,0), and then along the semicircle $y = -\sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$. (**Extra credit**: evaluate the line integral directly, to get the same answer.)
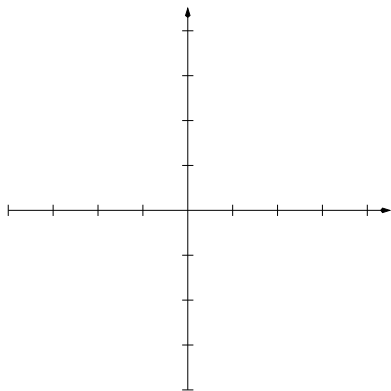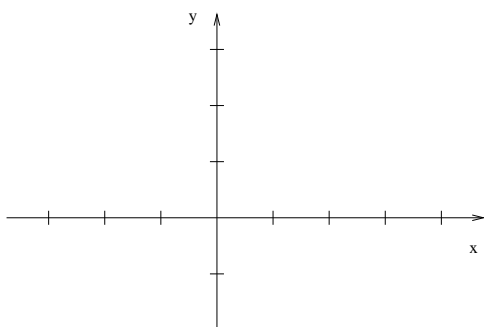
Problem 11 Use the divergence theorem to evaluate

$$I = \int_S \int \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x,y) = \langle x^2 y, -x^2 z, z^2 y \rangle$, and where S is the box bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$.

Problem 12 Evaluate the integral

$$I = \int_R \int (x+y) dA$$

where R is the square with vertices (0,0), (2,3), (5,1), and (3,-2), using the transformation $x = 2u + 3v$, $y = 3u - 2v$.

Problem 13 What is the value of the integral

$$\int_S \int x \, dS$$

where S is the portion of the paraboloid given by the equation $z = 1 - x^2 - y^2$ and located above the xy-plane?

Problem 14 Consider the vector function $\mathbf{F}(x, y) = \langle 3x^2, 2y \rangle$. Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the section of the parabola $y = x^2$ for $0 \leq x \leq 1$, in two ways: using the fundamental theorem of line integrals, and directly.