Section Summary: 14.4
Tangent Plane Approximation

a. Definitions

The **tangent plane** to the surface S at the point $P(x_0, y_0, z_0)$ is the plane containing both tangent lines of the x and y cross-sections. Suppose f has continuous partial derivatives. Those tangent lines are given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0)$$

and

$$z - z_0 = f_y(x_0, y_0)(y - y_0)$$

An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The function

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linearization** of f at (x_0, y_0).

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)(x - x_0) + f_y(a, b)(y - y_0) + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

For a function $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

Differentials give us approximate changes to a function in the neighborhood of a point, generally when dx and dy are small.

b. Theorems

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

c. Properties/Tricks/Hints/Etc.

d. Summary

This is a simple generalization of the tangent line, of course: in each cross-section, the tangent plane contains the tangent line (provided it exists). One of the interesting results is that differentiability in just two directions means that the function is differentiable (i.e., smooth).