Homework 4.4: Numerical Calculus

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pp. 172, 1acf, 2acf, 3acf, 7, 8, 11ab, 12, 15, 27, 28, 29

#1acf

1. Use the composite midpoint rule with 3 subintervals to approximate

(a) $\int_1^3 \ln(\sin(x))dx$ [S]

(b) $\int_5^7 \sqrt{x \cos x}\, dx$

(c) $\int_1^4 \frac{e^x \ln(x)}{x} dx$ [A]

(d) $\int_{10}^{13} \sqrt{1+\cos^2 x}\, dx$

(e) $\int_{\ln 3}^{\ln 7} \frac{e^x}{1+x} dx$ [A]

(f) $\int_0^1 \frac{x^2-1}{x^2+1} dx$

In[474]:=
```
compMidpoint[f_, a_, b_, n_] :=
 Module[{h = (b - a) / n, int},
  int = 0;
  For[i = 1 / 2, i ≤ n, i++,
   int = int + f[a + i h];
  ];
  int * h
 ]
```

1a.

```
In[475]:= f[x_] := Log[Sin[x]]
{a, b} = {1.0, 3.0};
compMidpoint[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[477]= -0.604014605941021

Out[478]= $-0.702330002837328 - 3.33066907387547 \times 10^{-16}\, i$

1c.

```
In[479]:= f[x_] := E^x Log[x] / x
{a, b} = {1.0, 4.0};
compMidpoint[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[481]= 17.5296173324835

Out[482]= 18.1436195116005

1f.

```
In[483]:= f[x_] := (x^2 - 1) / (x^2 + 1)
{a, b} = {0.0, 1.0};
compMidpoint[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[485]= -0.575424604932802

Out[486]= -0.570796326794897

#2acf

2. Redo question 1 using the composite trapezoidal rule. [S] [A]

3. Redo question 1 using the composite Simpson's rule. [S] [A]

```
In[487]:= compTrap[f_, a_, b_, n_] :=
 Module[{h = (b - a) / n, fa = f[a], fb = f[b], int},
  int = (fa + fb) / 2;
  For[i = 1, i ≤ n - 1, i++,
   int = int + f[a + i h];
  ];
  int * h
 ]
```

1a.

In[488]:=
```
f[x_] := Log[Sin[x]]
{a, b} = {1.0, 3.0};
compTrap[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[490]= -0.929469389751241

Out[491]= $-0.702330002837328 - 3.33066907387547 \times 10^{-16}\, i$

1c.

In[492]:=
```
f[x_] := E^x Log[x] / x
{a, b} = {1.0, 4.0};
compTrap[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[494]= 19.3773960369059

Out[495]= 18.1436195116005

1f.

In[496]:=
```
f[x_] := (x^2 - 1) / (x^2 + 1)
{a, b} = {0.0, 1.0};
compTrap[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[498]= -0.561538461538461

Out[499]= -0.570796326794897

#3acf

2. Redo question 1 using the composite trapezoidal rule. [S] [A]

3. Redo question 1 using the composite Simpson's rule. [S] [A]

Note that we can once again get off cheaply, by using two approximations, and averaging them together in a sensible way:

In[500]:=
```
compSimp[f_, a_, b_, n_] := (2 compMidpoint[f, a, b, n] + compTrap[f, a, b, n]) / 3
```

1a.

In[501]:=
```
f[x_] := Log[Sin[x]]
{a, b} = {1.0, 3.0};
compSimp[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[503]= -0.712499533877761

Out[504]= $-0.702330002837328 - 3.33066907387547 \times 10^{-16}\, i$

1c.

In[505]:=
```
f[x_] := E^x Log[x] / x
{a, b} = {1.0, 4.0};
compSimp[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[507]= 18.145543567291

Out[508]= 18.1436195116005

1f.

In[509]:=
```
f[x_] := (x^2 - 1) / (x^2 + 1)
{a, b} = {0.0, 1.0};
compSimp[f, a, b, 3]
Integrate[f[x], {x, a, b}]
```

Out[511]= -0.570795890468022

Out[512]= -0.570796326794897

#7 & 8

7. Use the (simple) trapezoidal rule on $\int_0^\pi \sin^4 x\,dx$ to help estimate the number of intervals $[0,\pi]$ must be divided into in order to approximate $\int_0^\pi \sin^4 x\,dx$ to within 10^{-4} using the *composite* trapezoidal rule. NOTE: $\int_0^\pi \sin^4 x\,dx = \frac{3}{8}\pi$. [𝕊]

8. Repeat question 7 using the midpoint rule. [𝔸]

#7: 109 will suffice (turns out 3 is enough) (Check -- with p. 294)

```
In[513]:= f[x_] := Sin[x]^4
          {a, b} = {0.0, Pi};
          true = Integrate[f[x], {x, 0, Pi}]
          estimate = compTrap[f, a, b, 1]
          Manipulate[
           {compTrap[f, a, b, n], compMidpoint[f, a, b, n], true * 1.0},
           {n, 1, 20, 1}
          ]
          Solve[Abs[true - estimate] / n^2 == 10^(-4), n]
```

Out[515]= $\frac{3\pi}{8}$

```
Out[516]= 0.
```

Out[517]=

```
n
{0.346573590279973, 0.405465108108164, 1.17809724509617}
```

```
Out[518]= {{n → -108.54018818374}, {n → 108.54018818374}}
```

#8: 141 will suffice (turns out 3 is enough)

```
In[519]:= estimate = compMidpoint[f, a, b, 1]
          Solve[Abs[true - estimate] / n^2 == 10^(-4), n]
Out[519]= 3.14159265358979
Out[520]= {{n → -140.124780409948}, {n → 140.124780409948}}
```

#11ab

11. Derive a summation formula for the composite version of

    (a) the midpoint rule.

    (b) Simpson's rule. [A]

    (c) Simpson's $\frac{3}{8}$ rule. [A]

    (d) the quadrature formula

$$\int_{x_0}^{x_0+h} f(x)\,dx \approx \frac{h}{2}\left[f\left(x_0+\frac{h}{3}\right)+f\left(x_0+\frac{2h}{3}\right)\right].$$

#12

12. Based on our discussion of composite integration, the error term for composite Simpson's rule applied to $\int_a^b f(x)\,dx$ with n subintervals is $O\left(\left(\frac{1}{n}\right)^4 f^{(4)}(\xi_n)\right)$. With a bit more work, it can be shown that the error term is actually $-\frac{b-a}{90}h^4 f^{(4)}(\xi_n)$ where $h = \frac{b-a}{n}$. No big-oh needed. This error is exact for some $\xi_n \in [a, b]$. Use this error term to find a theoretical bound on the error in estimating

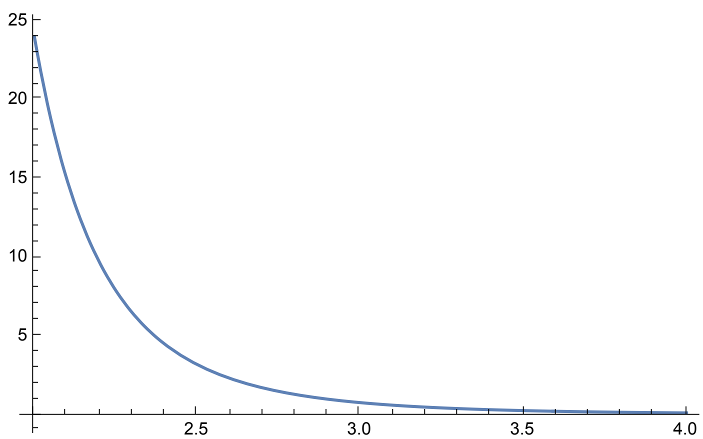
$$\int_2^4 \frac{1}{1-x}\,dx$$

using (composite) Simpson's rule with $h = 0.1$.

```
In[521]:= f[x_] := 1 / (1 - x)
f4[x_] = D[f[x], {x, 4}]
Plot[Abs[f4[x]], {x, 2, 4}, PlotRange → All]
Mbound = f4[2]
h = 0.1;
Abs[(4 - 2) / 90 h^4 Mbound]
```

Out[522]= $\frac{24}{(1-x)^5}$

Out[523]=



Out[524]= -24

Out[526]= 0.0000533333333333334

#15

15. Approximate $\int_1^3 \ln(\sin(x))dx$ using adaptive Simpson's method with tolerance 0.002. [S]

I'm starting with adaptive trapezoidal, because I've already got it in hand. I check it first against the author's example; then adapt it for Simpson's.

In[527]:=
```
aTraptiveQuad[f_, enda_, endb_, fenda_, fendb_, eps_, maxits_, verbose_] :=
 Module[{a = enda, b = endb, fa = fenda, fb = fendb, h, c, fc, err, int},
  h = (b - a) / 2;
  c = a + h;
  fc = N[f[c]]; (*
  (T2-T1)/3
   =
   h((fa+2fc +fb)/2-(fa+fb))/3
    =
    -h/6(fa-2fc+fb)*)
  err = h / 6 * (fa - 2 fc + fb);
  If[verbose, Print[{a, b, h (fa + 2 fc + fb) / 2, h (fa + fb), Abs[err]}]];
  If[Abs[err] < eps || maxits == 0
   , int = h (fa + 4 fc + fb) / 3;(* Simpson's! Ironic, no? *)
   ,
   int =
    aTraptiveQuad[f, a, c, fa, fc, eps / 2, maxits - 1, verbose]
     +
     aTraptiveQuad[f, c, b, fc, fb, eps / 2, maxits - 1, verbose]
  ];
  Return[int]
 ]
```

Check that it's working against the table of page 172.

Note that my approximation is a little better than the author's, because I compute Simpson's as my estimate rather than T2 -- since I've got everything I need to produce it.

a	b	$T_1(a,b)$	$T_2(a,b)$	$\frac{1}{3}\lvert T_2(a,b) - T_1(a,b)\rvert$	tol	
0	3	4.33555	4.42389	.02944	.00600	✗
0	1.5	1.95201	1.96732	.00510	.00300	✗
0	0.75	0.90763	**0.90997**	.00077	.00150	✓
0.75	1.5	1.05968	**1.06124**	.00051	.00150	✓
1.5	3	2.47187	**2.47961**	.00257	.00300	✓

$$\int_0^3 \ln(3+x)dx \approx 0.90997 + 1.06124 + 2.47961 = 4.45082$$

In[528]:=
```
f[x_] := Log[x + 3]
{a, b} = {0.0, 3.0};
eps = 0.006;
its = 30;
NIntegrate[f[x], {x, a, b}]
aTraptiveQuad[f, a, b, f[a], f[b], eps, its, True]
```

Out[532]= 4.45471994936401

```
{0., 3., 4.42389491358654, 4.33555763684425, 0.0294457589140959}
{0., 1.5, 1.96732551202838, 1.95201726408329, 0.00510274931503188}
{0., 0.75, 0.90996727074364, 0.907638048243911, 0.000776407499909815}
{0.75, 1.5, 1.06124348931223, 1.05968746378447, 0.000518675175918429}
{1.5, 3., 2.47960988220427, 2.47187764950325, 0.00257741090034197}
```

Out[533]= 4.45469313583631

In[534]:=
```
aSimptiveQuad[f_, enda_, endb_, fenda_, fmid_, fendb_, eps_, maxits_, verbose_] :=
 Module[{
   a = enda, b = endb, c = Mean[{enda, endb}],
   fa = fenda, fm = fmid, fb = fendb,
   fml, fmr,
   ends, mids, h, halfh, err, int},

  h = (b - a) / 2; (* NOTE: h is the half width of the interval. *)
  halfh = h / 2; (* NOTE: halfh is the half half-width of the interval!:) *)
  fml = f[a + halfh];
  fmr = f[b - halfh];
  (*
  s1=h/3(fa+4fm+fb);
  s2=h/6(fa+4(fml+fmr)+2fm+fb);
  err=(s2-s1)/15;
  *)
  (* These economize the calculations above: *)
  ends = fa + fb;
  mids = 4 (fml + fmr);
  err = (ends - mids + 6 fm) * h / 90;
  If[verbose, (* I have to actually do the calculations of s1 and s2: *)
   Print[{a, b, h / 3 (fa + 4 fm + fb), h / 6 (fa + 4 (fml + fmr) + 2 fm + fb), Abs[err]}]];

  If[Abs[err] < eps || maxits == 0, (* We've hit the end of the line: *)
   int = (ends + mids + 2 fm) h / 6, (* This is s2 *)
   int = (* Otherwise split divide the error in half, and split the interval: *)
    aSimptiveQuad[f, a, c, fa, fml, fm, eps / 2, maxits - 1, verbose]
     + (* not forgetting the reduce the number of iterations possible... *)
     aSimptiveQuad[f, c, b, fm, fmr, fb, eps / 2, maxits - 1, verbose]
  ];

  Return[int]
 ]
```

Our author means for you to go through the calculations, I guess, as one can see from the solution in the back of the book:

a	b	$S_1(a,b)$	$S_2(a,b)$	$\frac{1}{15}\lvert S_2(a,b)-S_1(a,b)\rvert$	tol	
1	3	−0.837026	−0.730741	0.00708	.002	✗
1	2	−0.046286	−0.045560	$4.8(10)^{-5}$	.001	✓
2	3	−0.684454	−0.661383	0.00153	.001	✗
2	2.5	−0.134349	−0.134243	$7.0(10)^{-6}$	.0005	✓
2.5	3	−0.527034	−0.523129	0.00026	.0005	✓

$$\int_1^3 \ln(\sin(x))dx \approx -0.045560 - 0.134243 - 0.523129 = -0.702932$$

```
In[535]:= f[x_] := Log[Sin[x]]
{a, b} = {1.0, 3.0};
eps = .002;
its = 30;
NIntegrate[f[x], {x, a, b}]
aSimptiveQuad[f, a, b, f[a], f[(a + b) / 2.0], f[b], eps, its, True]

Out[539]= -0.702330002837417

{1., 3., -0.837026840086607, -0.730741377417843, 0.00708569751125091}

{1., 2., -0.0462865678551063, -0.0455607083555639, 0.0000483906333028268}

{2., 3., -0.684454809562737, -0.661383920322872, 0.00153805928265767}

{2., 2.5, -0.134349753109684, -0.134243660715662, 7.07282626815384 × 10^-6}

{2.5, 3., -0.527034167213188, -0.523129874366284, 0.000260286189793605}

Out[540]= -0.702934243437509
```

#27

27. Write an Octave function that implements adaptive Simpson's rule as a recursive function. Some notes about the structure: [A]

   (a) The inputs to the function should be $f(x)$, a, b, and a maximum overall error, tol.

   (b) The output of the function should be the estimate and, if you are feeling particularly stirred, the number of function evaluations.

28. Use your code from question 27 to approximate $\int_1^3 \ln(\sin(x))dx$ with tolerance 0.002. [A]

29. Use your code from question 27 to approximate $$\int_0^1 \ln(x+1)dx$$ accurate to within 10^{-4}.

#28

```
In[541]:= f[x_] := Log[Sin[x]]
{a, b} = {1.0, 3.0};
eps = .002;
its = 30;
NIntegrate[f[x], {x, a, b}]
aTraptiveQuad[f, a, b, f[a], f[b], eps, its, True]
aSimptiveQuad[f, a, b, f[a], f[(a + b) / 2.0], f[b], eps, its, True]
```

```
Out[545]= -0.702330002837417

{1., 3., -1.16045722403475, -2.13074837587918, 0.323430383948143}
{1., 2., -0.0681757736868613, -0.133843391182126, 0.021889205831755}
{1., 1.5, -0.0349796597365324, -0.043777975615172, 0.00293277195954655}
{1., 1.25, -0.0269159190491789, -0.0281208042481096, 0.000401628399643574}
{1., 1.125, -0.0170553316667363, -0.0172154926043803, 0.0000533869792146677}
{1.125, 1.25, -0.00955836949832416, -0.00970042644479864, 0.0000473523154914901}
{1.25, 1.5, -0.0058409076950622, -0.00685885548842272, 0.000339315931120176}
{1.25, 1.375, -0.00434771409598859, -0.00447840795766201, 0.0000435646205578047}
{1.375, 1.5, -0.00123814917601127, -0.00136249973740019, 0.0000414501871296393}
{1.5, 2., -0.0162348149518559, -0.0243977980716893, 0.00272099437327779}
```

```
{1.5, 1.75, -0.00134945611567568, -0.00233147748195518, 0.0003273404554265}
{1.5, 1.625, -0.000126460156783001, -0.000248618449323819, 0.0000407194308469393}
{1.625, 1.75, -0.000977005593943295, -0.00110083766635186, 0.0000412773574695216}
{1.75, 2., -0.0128268804532439, -0.0139033374699007, 0.000358819005552262}
{1.75, 1.875, -0.00381699606175899, -0.0039465848381496, 0.0000431962587968696}
{1.875, 2., -0.0087400864278735, -0.00888029561509432, 0.0000467363957402741}
{2., 3., -0.769994565385209, -1.02661383285262, 0.0855397558224716}
{2., 2.5, -0.138790960708073, -0.152114583503239, 0.00444120759838864}
{2., 2.25, -0.0418939541120853, -0.0432522139901218, 0.000452753292678836}
{2., 2.125, -0.0159193371457057, -0.0160766133144597, 0.0000524253895846797}
{2.125, 2.25, -0.0256336173920864, -0.0258173407976255, 0.0000612411351796977}
{2.25, 2.5, -0.0934865316016792, -0.0955387467179509, 0.000684071705423902}
{2.25, 2.375, -0.0383169370903888, -0.0385419963604652, 0.0000750197566921282}
{2.375, 2.5, -0.0546522508338416, -0.054944535241214, 0.0000974281357908023}
{2.5, 3., -0.549745620880383, -0.61787998188197, 0.0227114536671955}
{2.5, 2.75, -0.180460002811799, -0.184574727209423, 0.00137157479920808}
{2.5, 2.625, -0.075762549744955, -0.0761722757234058, 0.000136575326150283}
{2.5, 2.5625, -0.0348346799459217, -0.0348811840015571, 0.0000155013518784713}
{2.5625, 2.625, -0.040825035547333, -0.0408813657433979, 0.0000187767320216353}
{2.625, 2.75, -0.103647617524066, -0.104287727088393, 0.000213369854775718}
{2.625, 2.6875, -0.0477247894436383, -0.0477950367917066, 0.0000234157826894416}
{2.6875, 2.75, -0.0557617312471123, -0.0558525807323596, 0.0000302831617490952}
{2.75, 3., -0.349323808183009, -0.36517089367096, 0.00528236182931682}
{2.75, 2.875, -0.142381037188527, -0.143570588158614, 0.000396516990029031}
{2.75, 2.8125, -0.0652753960948927, -0.0653985752970348, 0.0000410597340473849}
{2.8125, 2.875, -0.076804323084591, -0.0769824618914924, 0.0000593796023004775}
{2.875, 3., -0.202634933171133, -0.205753220024395, 0.00103942895108736}
{2.875, 2.9375, -0.0912806527219131, -0.0915637519163501, 0.0000943663981456874}
{2.875, 2.90625, -0.0435606981303071, -0.0435916835503102, 0.0000103284733343721}
{2.90625, 2.9375, -0.0476487175792166, -0.0476889691716029, 0.000013417197462101}
{2.9375, 3., -0.110546768128186, -0.111071181254783, 0.000174804375532414}
{2.9375, 2.96875, -0.0523906460258852, -0.052445169540131, 0.0000181745047486062}
{2.96875, 3., -0.0580234097581308, -0.0581015985880547, 0.0000260629433079593}
```

Out[546]= -0.702331484550436

```
{1., 3., -0.837026840086607, -0.730741377417843, 0.00708569751125091}
{1., 2., -0.0462865678551063, -0.0455607083555639, 0.0000483906333028268}
{2., 3., -0.684454809562737, -0.661383920322872, 0.00153805928265767}
```

$\{2., 2.5, -0.134349753109684, -0.134243660715662, 7.07282626815384\times 10^{-6}\}$

```
{2.5, 3., -0.527034167213188, -0.523129874366284, 0.000260286189793605}
```

Out[547]= -0.702934243437509

#29

In[548]:=
```
f[x_] := Log[x + 1]
{a, b} = {0.0, 1.0};
eps = 10 ^ (-4.0);
its = 30;
NIntegrate[f[x], {x, a, b}]
aTraptiveQuad[f, a, b, f[a], f[b], eps, its, False]
aSimptiveQuad[f, a, b, f[a], f[(a + b) / 2.0], f[b], eps, its, False]
```

Out[552]= 0.386294361119891

Out[553]= 0.386294342100538

Out[554]= 0.386259562814567