Lufkin's Log Rule, Circa 1953: Estimating Board Feet of Lumber

Math Modeling Class, Spring, 2020

February 8, 2020

Abstract

Presented with a mysterious wooden measuring device[1], the class discovered that it is a tool for estimating board feet of lumber from a log(or a standing tree).

Lettering on the device identified it as having been manufactured in Barrie, Ontario, Canada, by the Lufkin Rule Company. A comparison with similar devices discovered on the internet suggests that it was manufactured circa 1953 (the Barrie plant opened in 1948, according to current owners Apex Tool Group[2]).

A study of similar log rules revealed that there are various methods by which one can estimate the board feet; however, an application of linear regression on the markings permitted us to determine that the method used is the Ontario Log Rule.

1 Introduction and Data

1.1 History

In the fall of 2019 Andy Long's friend Dave Wilkins presented Andy with this stick (Figure 1). Dave wasn't sure just exactly what it is, but thought that Andy (as a mathematician) would appreciate it. When Andy learned that he would be teaching math modeling in the spring, he realized that this would make for an interesting introduction to linear regression.

What follows is a summary of methods and results which have permitted us to reach the conclusion that this tool is a log rule, marked according to the Ontario Log Rule.

1.2 Data

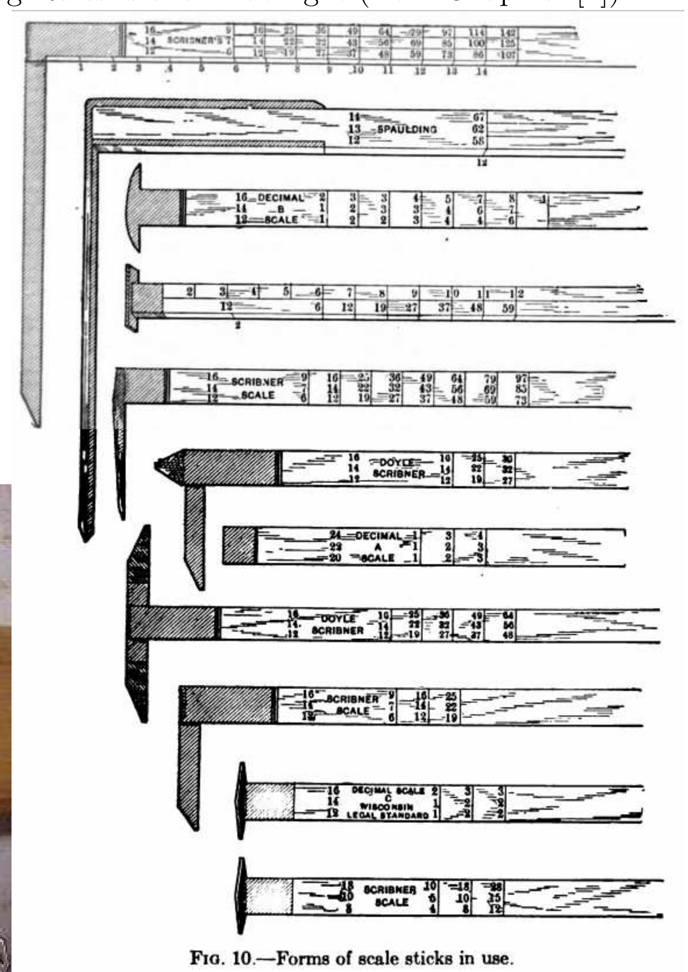
The data are markings on the rule, many of them difficult to read or completely obscured. The class's mission was to "recreate" the stick's markings, exactly – that is, to determine the function(s) that give rise to the data, so that we could restore the markings (or create a more legible version of the stick).

Additional "data" came in the form of a label, which was considerably obscured on the rule: a clearer version of that embossed label is shown in Figure 2.

[1]Thanks to Dave Wilkins.

Figure 1: Dave's Mystery Stick. We scanned in one "working side" (the 10 foot and 12 foot log measure side). On the other side were the markings for the 14 and 16 foot logs, and on the edge was another set of markings (for the 8 foot logs, each edge representing board feet at inch intervals, staggered by a half-inch). It was considered "an innovation in the application of the Ontario Log Rule" that values were given at half-inch classes[7].

Figure 2: A label like the one at left appeared on our rule, although ours was highly obscured by time and hard use. A variety of typical Log Rules is shown at right (from Chapman[4]).



2 Methods

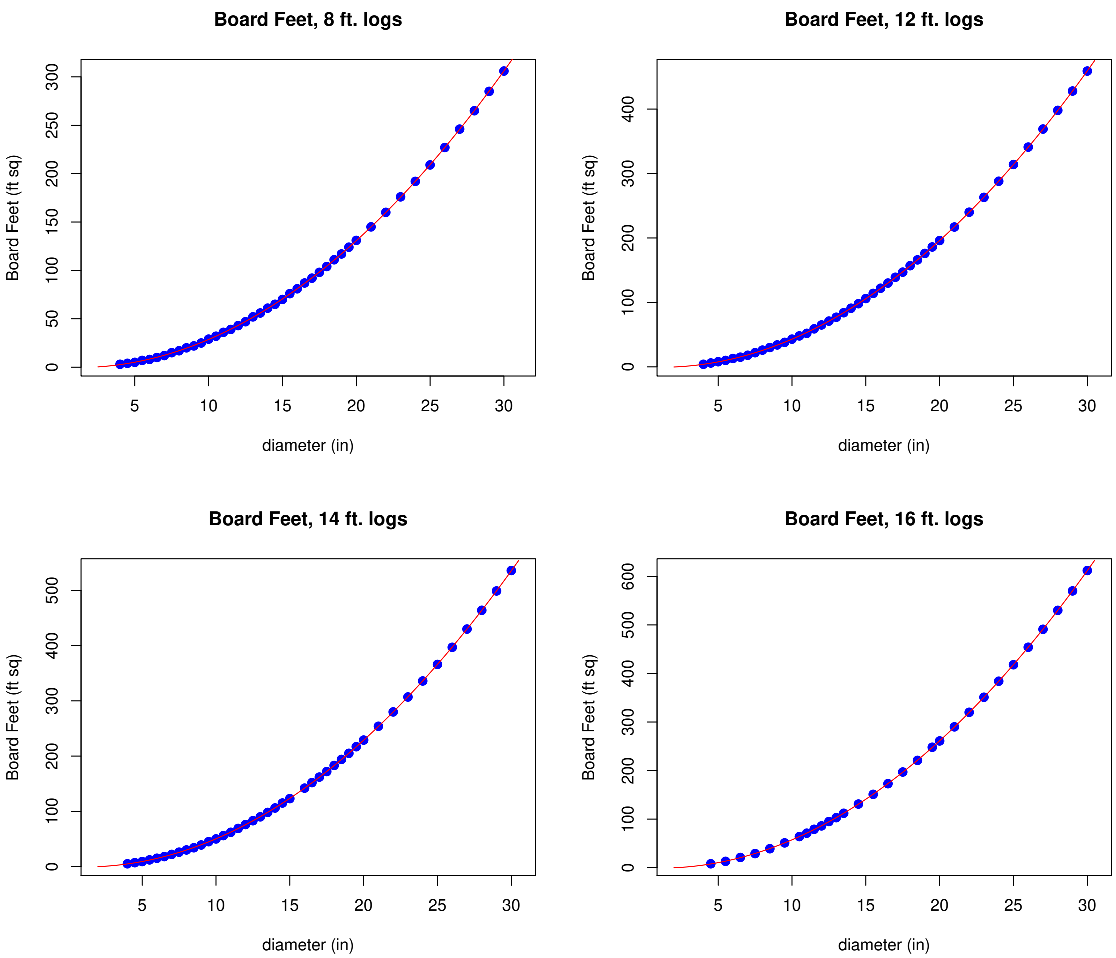
In math modeling the first step (once one has the data) is to begin to summarize and visualize the data. In this section we illustrate our methods of visualization, and then demonstrate how we reach our conclusions based on analysis of the data.

2.1 Graphs of the Data

We read every value we could from the tool, although in many cases it was hard to decipher exactly what the marking was. We hoped that any errors we introduced through our inability to be sure of markings would be smoothed out by the regression process – and that we could later repair the data, if we could discover the secret to the markings.

With that in mind, we present in Figure 3 the data for which we could create a relatively complete data set: data for four log lengths, 8ft, 12ft, 14ft, and 16ft. The 10ft markings were pretty sporadic, and were not used for analysis. However a few markings were visible, and these allowed us to confirm that the reconstructed 10ft marks were correct using the Ontario Log Rule.

Figure 3: These four plots illustrate the data (such as we could make out) for four of the five log-lengths marked on the tool. In addition, we include the final model, illustrating the beautiful fit provided by the Ontario Log Rule.



2.2 Data Analysis: Identification of Marking Pattern

There are many different Board Feet log rules[3, 6], so one might have to sort through to see which is most appropriate. There is one nice on-line calculator, which illustrates many of the common rules simultaneously for a given log diameter and length[9]: this provides a quick means of eliminating most of the methods for this tool. The website issues this interesting warning to the user: "*(Not very scientific, serves no real world purpose and should not be taken as an accurate method of calculating board feet content of a log.)*"

The Ontario Log Rule[10, 8] is given by the following:

$$BF = \frac{L}{12}(0.55D^2 - 1.2D)$$

where BF is board-feet (in square feet), D is the diameter of the smaller end (inside bark, in inches), and L is the length of log (in feet). The Ontario Rule was only developed in 1952, and adopted by Ontario in 1953[7]. Note that BF is linear in L, the length of the log: if we double the length of the log, we should double the number of board feet.

The Ontario Log Rule is derived from structural considerations, a "general formula for lumber recovery" (L.R., developed by H. H. Chapman)[7]:

$$L.R. = \left((1-b)\frac{\pi D^2}{4} - AD\right)\frac{L}{12}$$

where

- b is percentage of wood volume deducted for saw kerf, shrinkage and sawing inaccuracy;
- D is the (top) diameter of the log (inches);
- A is the thickness (inches) of a plank, representing the allowance for slags and edgings; and
- L is the length of the log (feet).

Values for $b = 0.30$ and $A = 1.2$ were chosen, at which point the formula became established as the "Ontario Log Rule".

We include a figure (Figure 4) from Chapman's 1921 paper "The measurement of logs and the construction of log rules"[5], in which Chapman illustrates the process of estimating board feet of lumber from a log.

2.3 Linear Regression Results

Our suspicion that the rule used to create all the markings on this particular tool is the Ontario Log Rule were verified by linear regression.

If we regress the 12-foot values against a quadratic in D, then we should get parameter estimates for D of -1.2, and for D^2 of .55 (per the Ontario Rule). Furthermore, the constant term should not be significant. When we do so, we obtain

```
Linear Regression:        Estimate            SE              Prob

Constant                 0.106504       (0.255070)          0.67851
inches                   -1.21898       (3.461117E-2)       0.00000
inches^2                 0.550846       (1.030239E-3)       0.00000
```

Figure 4: This figure appears in Chapman's 1921 paper *The measurement of logs and the construction of log rules*. It illustrates the waste in sawing a log into lumber.

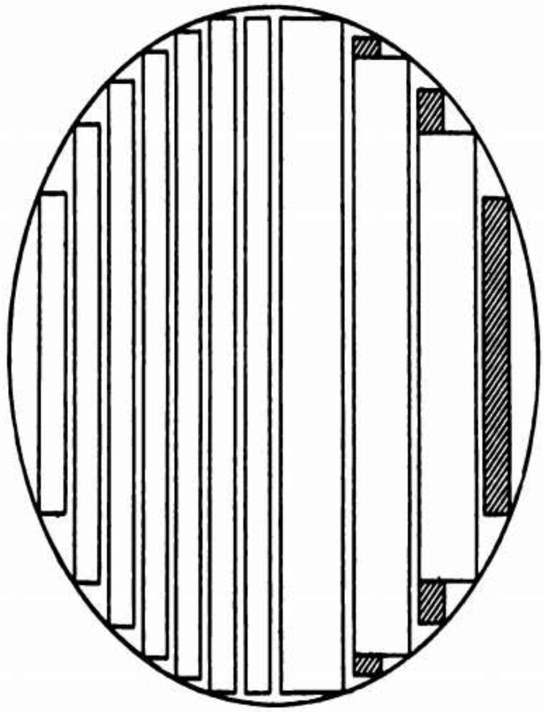


FIGURE 1.

The left half of the figure represents the so-called diagram method of constructing a log rule by plotting the 1-inch boards which may be sawed from logs of different diameters. On the right side is shown the affect of cutting 2'4-inch plank from small logs. The sawing in saw kerf is offset by a loss of 1-inch lumber.

```
R Squared:                0.999993
Sigma hat:                0.353536
Number of cases:                43
Degrees of freedom:             40
```

For other cases, the parameter estimates should be $\frac{L}{12}$ times these parameter values (Table 1), and the constant term should not be significantly different from zero.

Table 1: Expected parameters in a regression model of the form $aD^2 + bD + c$, provided the rule is the Ontario Log Rule.

Length	D^2 term	D term	constant term
8ft	0.367	-0.8	0
12ft	0.550	-1.2	0
14ft	0.642	-1.4	0
16ft	0.733	-1.6	0

The rest of the regression results follow: note that the constant terms are not significant, and that the regression parameters match the coefficients in the table.

```
Eights:

Constant                 -3.297308E-3   (0.225538)         0.98841
inches                   -0.804102      (3.060388E-2)      0.00000
inches^2                 0.366869       (9.109580E-4)      0.00000

R Squared:               0.999987
Sigma hat:               0.312603
Number of cases:                43
Degrees of freedom:             40

Fourteens:

Constant                 0.208001       (0.196673)         0.29675
inches                   -1.41416       (2.685213E-2)      0.00000
inches^2                 0.642053       (7.995300E-4)      0.00000

R Squared:               0.999997
Sigma hat:               0.271043
Number of cases:                42
Degrees of freedom:             39

Sixteens:

Constant                 -3.567793E-2   (0.294023)         0.90432
inches                   -1.59359       (3.763398E-2)      0.00000
inches^2                 0.732947       (1.064040E-3)      0.00000

R Squared:               0.999998
Sigma hat:               0.293119
Number of cases:                30
Degrees of freedom:             27
```

As is typically done when parameter values are not statistically significant (not significantly different from 0), we eliminate those parameters and refit the model. This was done using R, with the following results for parameter values (Table 2 – compare with Table 1, and we note how astonishingly close the results are to those expected of the Ontario Log Rule).

Table 2: Parameters obtained using R in a regression model of the form $aD^2 + bD$ (without intercept term).

Length	D^2 term	D term
8ft	0.3669	-0.8045
12ft	0.5505	-1.2053
14ft	0.6413	-1.3873
16ft	0.7331	-1.5979

As an example of the output of one of our regressions, we include the 95% confidence intervals for the parameter estimates:

```
Coefficients:
      x   I(x^2)
-1.2053   0.5505

           2.5 %     97.5 %
x      -1.227726 -1.1828869
I(x^2)  0.549458  0.5514852
```

This says that, with 95% confidence, the true parameter values lie within those intervals; and it's comforting that the Ontario Log Rule values are within those intervals.

3 Conclusion

After considering the possible ensemble of available rules, we discover – based on linear regression – that the true rule used is the Ontario rule. This confirms our suspicions, since the word "Ontario" is stamped at the base of the log rule tool....

Once we know the rule, we can easily reconstruct the missing markings on the tool, which was one of our objectives. The reconstructions are given in the appendix.

Case closed! It's an Ontario Log Rule-based tool for producing estimates of the board feet of lumber in a log.

4 Appendix

4.1 Reconstruction

Table 3 gives the reconstruction of the stick, based on the legible data and the Ontario Log Rule.

4.2 Other Rules

This Table (Figure 5) is taken from the work by Ker[7].

4.3 Structural Rules

H. H. Chapman, who wrote the book on log measurement ("Forest Mensuration"[4]), tells us that "... American log rules were made by the practical mill men who were not versed in the use of graphic or formula methods of obtaining average values. The results were in some cases good, in other, very bad. The most sensible and in every respect the soundest and best method of making a log rule ever devised was that of plotting the dimensions of 1-inch boards on circles drawn to the size of logs of different diameters, and then computing the board foot contents from the diagram. In this way one of the earliest and best of the old log rules, the old Scribner rule, was made."[5]

Chapman describes how rules are typically quadratic, and gives some of the more common in his time (Figure 6). Unfortunately, the version of Scribner's given here does not correlate to the one in a more recent publication[3]: $(.79D^2 - 2D - 4)L/16$; and we've used the more recent one. Chapman appears to make a mistake in a few more formula following this one, so we are suspicious of his formula. On the other hand, names change and evolve (e.g. the "Doyle" and "Ontario" are sometimes confused – sensibly enough, since Doyle was the previously established rule for Ontario).

Scribner's Decimal C rule is based on geometric considerations of how many planks one can actually obtain from a given log[1]. As illustrated in Figure 7, the planks may be uniform in length; however that is not necessary, as shown earlier in this paper on the left hand side of Figure 4 (where the planks get steadily larger as the saw moves from the outside to the center of the log).

We construct our own rule, the MAT375 rule, using the variable length strategy of Figure 8: the longest plank will come from dead center; then a pair of planks, each of somewhat shorter length, will come from the right and left halves, etc.

We begin by assuming no kerf. If the log is of radius r, then the widths of the planks

Table 3: Reconstructed values for the log rule, using (rounded) values of the Ontario Log Rule. So "8ft" represents data, and "8ftR" is our reconstruction, based on the Ontario rule. The 10 foot markings were so poorly represented that we did not perform regression on them. However an inspection of the stick shows that the reconstructed values (10ftR) coincide with those few marks legible on the stick (next to the 12 foot markings). A value of "nil" indicates that the marking was illegible or missing.

8ft	8ftR	10ftR	12ft	12ftR	14ft	14ftR	16ft	16ftR
nil	0	0	nil	0	nil	0	nil	0
nil	1	0	nil	0	nil	1	nil	1
nil	2	1	nil	1	nil	2	nil	2
3	3	2	nil	3	nil	3	nil	3
4	4	3	4	4	5	5	nil	5
5	5	5	6	6	7	7	8	8
7	7	6	8	8	9	9	nil	10
8	8	8	10	10	12	12	13	13
10	10	11	13	13	15	15	nil	17
12	12	13	15	15	18	18	21	21
15	15	15	18	19	22	22	nil	25
17	17	18	22	22	26	26	29	29
20	20	21	26	26	30	30	nil	34
22	23	25	30	30	34	34	39	39
25	25	28	34	34	39	39	nil	45
29	29	32	38	38	45	45	51	51
32	32	36	43	43	50	50	nil	57
36	36	40	48	48	56	56	64	64
39	39	44	52	53	62	62	71	71
43	43	49	59	59	69	69	79	79
47	47	54	65	65	76	76	86	86
52	52	59	71	71	83	83	95	95
56	56	64	77	77	90	90	103	103
61	61	70	84	84	98	98	112	112
65	65	76	91	91	106	106	nil	121
70	71	82	98	98	115	115	131	131
76	76	88	106	106	123	123	nil	141
81	81	95	114	114	nil	132	151	151
87	87	101	122	122	142	142	nil	162
92	92	108	130	130	152	152	173	173
98	98	115	139	139	162	162	nil	185
104	104	123	147	147	172	172	197	197
111	111	131	157	157	183	183	nil	209
117	117	138	166	166	194	194	221	221
124	124	146	176	176	205	205	nil	234
131	131	155	186	186	217	217	248	248
nil	138	163	196	196	229	229	261	261
145	145	172	nil	207	nil	241	nil	275
nil	152	181	217	217	254	254	290	290
160	160	190	nil	228	nil	267	nil	305
nil	168	200	240	240	280	280	320	320
176	176	210	nil	251	nil	293	nil	335
nil	184	219	263	263	307	307	351	351
192	192	230	nil	276	nil	321	nil	367
nil	200	240	288	288	336	336	384	384
209	209	251	nil	301	nil	351	nil	401
nil	218	261	314	314	366	366	418	418
227	227	273	nil	327	nil	382	nil	436
nil	236	284	341	341	397	397	454	454
246	246	295	nil	354	nil	414	nil	473
nil	255	307	369	369	430	430	491	491
265	265	319	nil	383	nil	447	nil	511
nil	275	331	398	398	464	464	530	530
285	285	344	nil	413	nil	481	nil	550
nil	295	356	428	428	499	499	570	570
306	306	369	nil	443	nil	517	nil	591
nil	317	383	459	459	536	536	612	612
nil	328	396	nil	475	nil	554	nil	633

will be

$$D\sqrt{1-\left(\frac{1}{D}\right)^2}$$

$$D\sqrt{1-\left(\frac{3}{D}\right)^2}$$

$$\vdots$$

$$D\sqrt{1-\left(\frac{(2n-1)}{D}\right)^2}$$

Figure 5: This table illustrates the comparative differences between a variety of rules popular in different times and places.

TABLE 1

BOARD-FOOT CONTENT OF 16-FOOT SAWLOGS AS ESTIMATED BY VARIOUS LOG RULES

				Log Rule Volume, board feet					
Top D.I.B., in.	Doyle	Ont.	B.C.	Scribner Decimal C	Scribner Formula Rule	Alberta Modified Int'l 5/16"	Int'l 5/16" Rule Rounded	Quebec (Roy)	N.B.
4	0	5	5				5	7	9
6	4	17	15	20	12	18	20	20	20
8	16	34	32	30	31	37	40	39	40
10	36	57	54	60	55	62	65	65	64
12	64	86	84	80	86	93	95	97	96
14	100	121	118	110	123	129	135	135	130
16	144	162	160	160	166	172	180	180	170
18	196	209	207	210	216	222	230	231	229
20	256	261	261	280	272	277	290	289	300
22	324	320	319	330	334	338	355	353	362
24	400	384	384	400	403	405	425	423	432
26	484	454	455	500	478	479	500	500	507
28	576	530	531	580	559	558	585	583	614
30	676	612	611	660	647	644	675	673	706

It is of interest to note (See Table 1) that the Ontario Log Rule, developed in 1952, yields tabular values almost identical with those given by the British Columbia Log Rule, constructed in 1894, for logs eight to eighteen feet in length. However, the practice of dropping fractions of inches would lower somewhat scaled contents by the Ontario Log Rule.

Figure 6: Some of the rules common at the time of Chapman's "Forest Mensuration"[4], expressed as quadratics.

For the following rules, the formulæ read:

Doyle, $$\text{B.M.} = (.75D^2 - 6D + 12)\frac{L}{12},$$

Scribner, $$\text{B.M.} = (.555D^2 - .55D - 23)\frac{L}{12};$$

Maine, $$\text{B.M.} = (.635D^2 - 1.45D + 2)\frac{L}{12};$$

Champlain, $$\text{B.M.} = (.62832D^2 - D)\frac{L}{12};$$

Vermont, $$\text{B.M.} = (.50D^2)\frac{L}{12}.$$

Figure 7: Structural rules such as Scribner's Decimal C Rule are based on geometric considerations of how many planks one can actually obtain from a given log. (From a U.S. Department of Agriculture publication[1]).

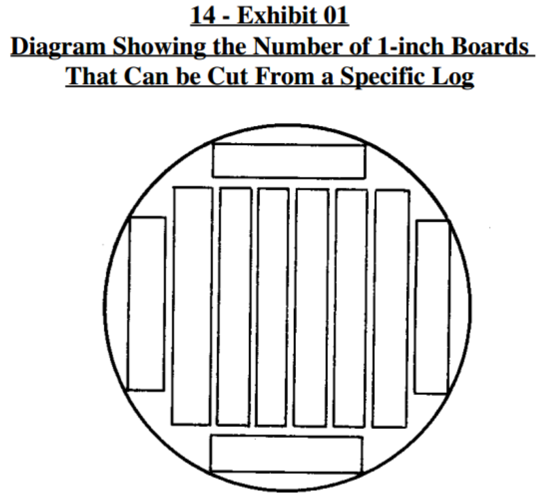
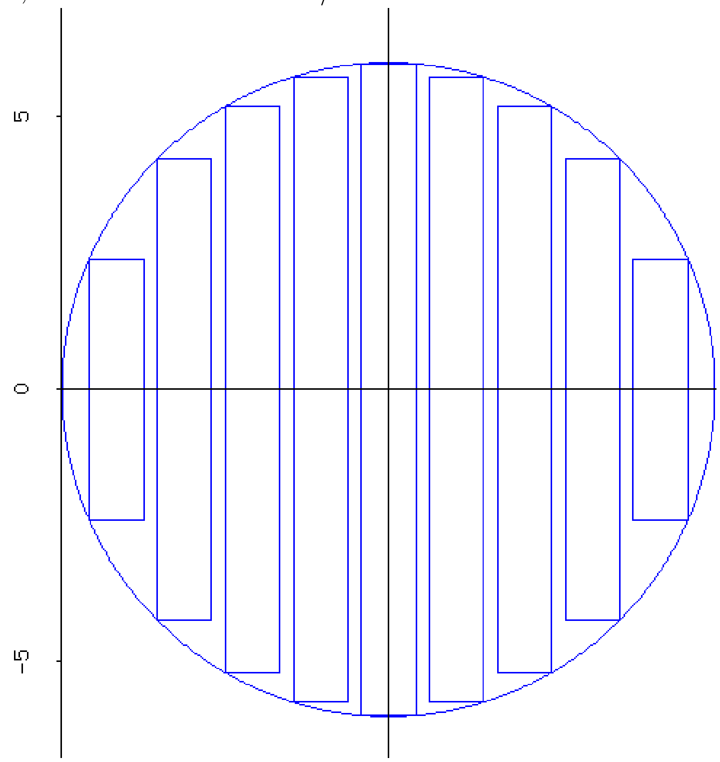


Figure 8: Our class rule is designed to maximize the number of 1 inch thick planks from a log. Because we have a kerf, some of the log is lost as sawdust. This figure illustrates a log of 12 inch diameter, and a saw blade of 1/4 inch kerf.



while $(2n-1)/2 < r$. We double the planks for all but the center plank, and so

$$BF_0 = LD\left[\sqrt{1-\left(\frac{1}{D}\right)^2} + 2\sum_{n=2}^{Floor\left(\frac{D+1}{2}\right)}\sqrt{1-\left(\frac{(2n-1)}{D}\right)^2}\right]$$

If we add in a positive kerf k, then we uglify the equation above as follows:

$$BF_k = LD\left[\sqrt{1-\left(\frac{1}{D}\right)^2}+2\sum_{n=2}^{Floor\left(\frac{D+1+2k}{2+2k}\right)}\sqrt{1-\left(\frac{(2n-1)+(2n-2)k}{D}\right)^2}\right]$$

A quick check shows that the formula BF_k reduces to the model BF_0 when $kerf = 0$.

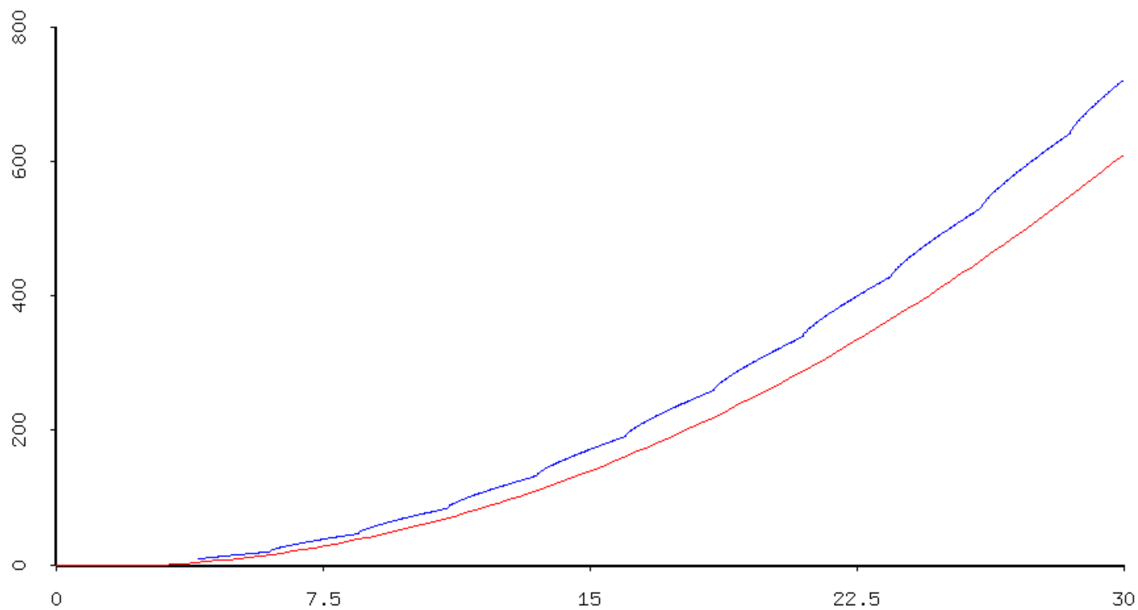
This model illustrates a number of important properties described by Chapman[5]. The percent loss from sawdust is

$$L_{sawdust} \approx \frac{100k}{1+k}$$

and tends to this value asymptotically (so that a kerf of 1/4 inch means a loss of 20%). And the loss from slag, L_{slag} goes asymptically to 0 as the diameter gets large.

Our structural model has the unusual property of being non-differentiable. An example of the model for a given log length (16 foot logs) is given in Figure 9.

Figure 9: A graph (in blue) of the estimates provided by the MAT375 rule of board feet of lumber, with a 1/4 inch kerf. Also plotted is the Ontario Rule, which shows that the 1/4 model suggests a much larger quantity of lumber.

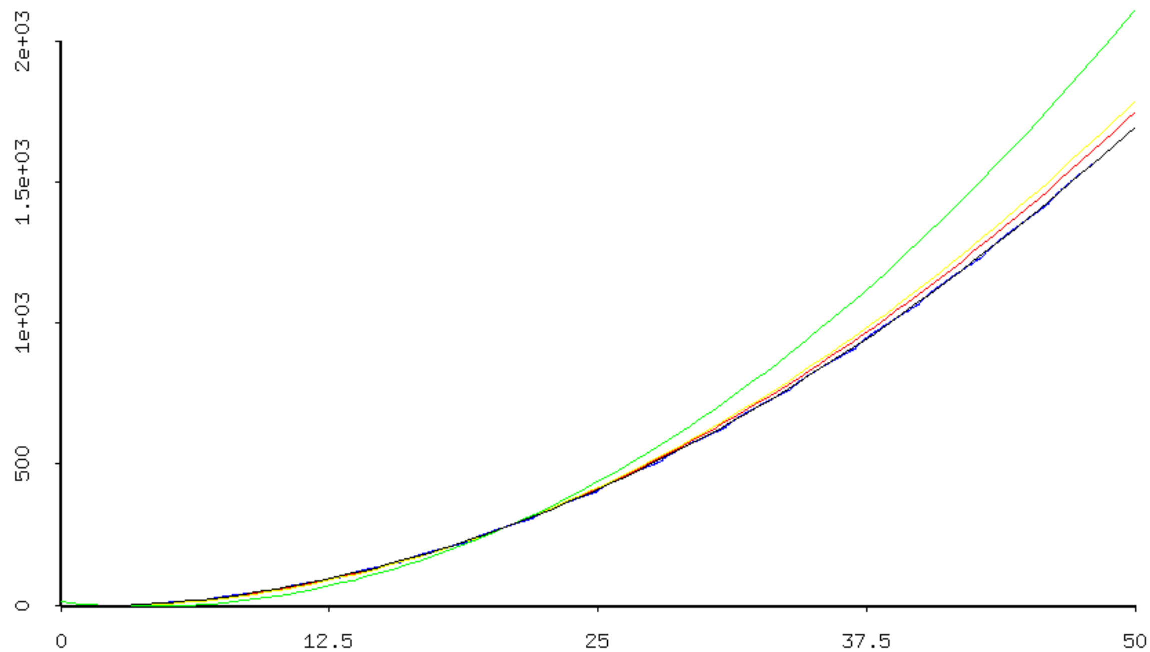


The model with kerf of 1/2 inch is quite similar to the Ontario Rule, as shown in Figure 10. We can replace the rule with a quadratic model: $BF_{\frac{1}{2}} \approx \frac{L}{12}(0.52D^2 - 0.67D)$, which was obtained by linear regression (shown below).

The regression equation we created as a good approximation to the MAT375 Rule is obtained via the regression below:

Linear Regression:	Estimate	SE	Prob
Linear Coef	-0.890759	(8.037547E-3)	0.00000
Quadratic Coef	0.698162	(8.642692E-5)	0.00000

Figure 10: Here is the MAT375 Rule with 1/2 inch kerf, showing how closely it resembles the Ontario and BC rules (the Doyle Rule is far off). The MAT375 Rule (in blue) fits very well with two quadratics (the Ontario and British Columbia Rules). We also fit our model with a quadratic, so that is easier to compute. The quadratic model is $\frac{L}{12}(0.52D^2 - 0.67D)$, and was obtained by regression.




```
R Squared:                  0.999998
Sigma hat:                  4.477420
Number of cases:                1000
Degrees of freedom:              998
```

We regressed without an intercept term, because in our initial regression it was not significantly different from 0.

References

[1] Anonymous. National forest log scaling handbook. Technical Report FSH 2409.11, Forest Service, U.S. Department of Agriculture, October 2006.

[2] Apex Tool Group. Lufkin(R) – 1948 Barrie, Ontario, Feb 2020. http://www.apextoolgroup.com/content/lufkin-1948-barrie-ontario.

[3] D. Cassens. Log and tree scaling techniques. Technical Report FNR-191, Forestry and Natural Resources, Purdue University, Dec 2001. https://www.extension.purdue.edu/extmedia/FNR/FNR-191.pdf.

[4] H. H. Chapman. *Forest Mensuration.* John Wiley & sons, Incorporated, 1921.

Table 4: This table compares a variety of rules (Scribner[3] is also included, as it is officially the National Forest Service's rule: "The Scribner Decimal C Log Rule, the International *-1/4-Inch log rule, or the Smalian cubic volume rule as used by the Forest Service are authorized under 36 CFR 223.3 for uniform scaling of sawtimber."[1]).

diameter	Doyle	Ontario	BC	Scribner	$BF_{1/2}$	$BF_{1/4}$	BF_0
4	0	5	5	1	5	10	12
6	4	17	15	12	20	21	31
8	16	34	32	31	39	44	57
10	36	57	55	55	57	74	92
12	64	86	84	86	90	110	136
14	100	121	119	123	127	151	188
16	144	162	160	166	159	193	248
18	196	209	207	216	210	250	317
20	256	261	261	272	264	314	394
22	324	320	320	334	312	384	479
24	400	384	386	403	381	459	573
26	484	454	457	478	452	532	675
28	576	530	535	559	515	625	785
30	676	612	619	647	601	723	904
32	784	700	708	741	690	827	1032
34	900	793	804	841	769	935	1167
36	1024	893	906	948	873	1040	1311
38	1156	998	1015	1061	978	1166	1463
40	1296	1109	1129	1180	1073	1299	1624
42	1444	1226	1249	1306	1194	1436	1793
44	1600	1349	1376	1437	1316	1578	1971
46	1764	1478	1508	1576	1427	1715	2157
48	1936	1613	1647	1720	1566	1876	2351
50	2116	1753	1791	1871	1705	2042	2554
52	2304	1900	1942	2028	1832	2213	2765
54	2500	2052	2099	2192	1988	2389	2984
56	2704	2210	2262	2361	2144	2558	3212
58	2916	2374	2431	2538	2287	2753	3448

[5] H. H. Chapman. The measurement of logs and the construction of log rules. *The Southern Lumberman*, pages 114–117, 1921.

[6] F. Freese. A collection of log rules. General Technical Report FPL: 1974-754-554/61, U.S.D.A. Forest Service, 1974. https://library.dbca.wa.gov.au/static/FullTextFiles/954888.pdf.

[7] J. W. Ker. The measurement of forest products in Canada: past, present and future historical and legislative background. *The Forestry Chronicle*, pages 29–38, March 1966. https://pubs.cif-ifc.org/doi/pdf/10.5558/tfc42029-1.

[8] Ontario Woodlot Association. Log scaling - Ontario log rule, Jan 2003. https://www.ontariowoodlot.com/publications/owa-publications/woodland-

notes/download/70_d99a020e494a3971380bdc26f2a56244.

[9] VMnet. Comparing results from various board feet log rules, Jan 2020. https://www.spikevm.com/calculators/logging/BF-Log-Rules.php.

[10] VMnet. Ontario log rule, Jan 2020. https://www.spikevm.com/calculators/logging/ontario-log-rule.php.