LAB #10

SIR Model of a Disease

Goal: Model a disease and investigate its spread under certain conditions. Use graphs generated by ***pplane*** (and its many options) to estimate various quantities.

Required tools: MATLAB routine ***pplane*** and its graphing options.

Discussion

The ***SIR*** model is a mathematical model of the spread of an infectious disease satisfying the following assumptions:

(i) the disease is short lived and rarely fatal

(ii) the disease is spread by contact between individuals

(iii) individuals who recover develop immunity.

If $S(t), I(t)$ and $R(t)$ represent the number of **S**usceptible, **I**nfected and **R**ecovered individuals in a population, then it can be shown that with the above assumptions, the system of equations modeling this disease satisfies:

$$\begin{cases} \dfrac{dS}{dt} = -aSI \\[2ex] \dfrac{dI}{dt} = aSI - bI \qquad\qquad (*) \\[2ex] \dfrac{dR}{dt} = bI \end{cases}$$

where a and b are positive constants.

Because the total population N remains constant (at least in the short term), for small values of t, we have

$$N = S(t) + I(t) + R(t) \qquad (**)$$

If we know $S(t)$ and $I(t)$, we also know $R(t)$. Hence the 3^{rd} equation in $(*)$ is not necessary and since the system is also autonomous ***pplane*** can be used.

Assignment

Throughout we shall assume that the total population is $N = 5$ million people, t is measured in months, and $0 \leq t \leq 4$.

(1) Experimentally it was determined that for a certain strain of flu spreading through this community that it satisfies the SIR model with $\boldsymbol{a} = \mathbf{1.1}$ and $\boldsymbol{b} = \mathbf{1.3}$. Initially there are 1 million infected and there are 3 million who do not have, nor have ever had, this particular flu virus.

(a) Use ***pplane*** to plot the phase portrait (in the SI-plane) with the values of a and b and initial conditions as above (use the options "Keyboard Input" and "Specify a computation interval"). Print out your plot. Describe in words what is happening to S and I as time t progresses from 0 to 4 months.

(b) From your graphs and the various graphing options in ***pplane*** , approximate the maximum number of infected people during the first 4 months. Estimate when this occurs. (You can adjust your "Specify a computation interval" to get better estimates.)

(c) Is the number of susceptibles ever the same as the number of infected people when $0 \le t \le 4$? If so, estimate when this occurs. Print out your graph of both S and I on a single plot.

(d) Estimate $S(2), I(2)$, and $I(3)$. These values will be used in the next part.

(2) From 1(d), you estimated $S(2), I(2), I(3)$. Suppose that at $t = 2$ the flu virus mutates so that the value of a changes to $\boldsymbol{a} = \mathbf{0.5}$ ($\boldsymbol{b}$ remains the same at $\boldsymbol{b} = \mathbf{1.3}$). Estimate I, the number of infected, 1 month after the virus mutates. Compare this to the value of $I(3)$ from above. Explain the difference.

(3) Local officials will consider various strains of flu an epidemic when the number of infected reaches a maximum. Using the model $(*)$ in ***pplane*** , with the values shown in the tables below, estimate the values of S at the time t^* when the number of infected is a maximum (don't submit plots):

$\boldsymbol{a} = \mathbf{1.5},\ \boldsymbol{b} = \mathbf{3}$

$S(0)$	$I(0)$	$S(t^*)$
3	2.0	
3	1.1	
3	0.5	
2	3	

$\boldsymbol{a} = \mathbf{2},\ \boldsymbol{b} = \mathbf{3}$

$S(0)$	$I(0)$	$S(t^*)$
3	2.0	
3	1.1	
3	0.5	
2	3	

For the system $(*)$, explain why you should expect that $I(t)$ reaches a maximum when $S(t^*) = \dfrac{b}{a}$. Do the results in your tables above provide numerical evidence of the above statement ?

(4) It is sometimes of interest to rescale or normalize the quantities $\boldsymbol{S}, \boldsymbol{I}, \boldsymbol{R}$ in the model. If N is the total population, let

$$\boldsymbol{s} = \frac{\boldsymbol{S}}{\boldsymbol{N}} \qquad \boldsymbol{i} = \frac{\boldsymbol{I}}{\boldsymbol{N}} \qquad \boldsymbol{r} = \frac{\boldsymbol{R}}{\boldsymbol{N}},$$

which represents the fraction of total population which are susceptible, infected and recovered, respectively. What is $\boldsymbol{s} + \boldsymbol{i} + \boldsymbol{r}$?

(a) Using (∗), derive the corresponding system for the new variables $\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{r}$.

(b) An "unnamed source" in the local government made several statements to the press (shown below) concerning the strain of flu modeled by (∗), with $\boldsymbol{a} = \mathbf{1.1}$, $\boldsymbol{b} = \mathbf{1.3}$ and $N = 5$. Your job is to use ***pplane*** to determine whether his statements are accurate

Statement #1: *If initially half the population is susceptible but only 10% of the population is infected, then there will never be more than 15% of the population infected.*

Statement #2: *If initially* 25% *of the population is infected and 25% is susceptible, then the number of susceptibles is never again the same as the number of infected.*

Statement #3: *If initially there are no individuals with immunity and the number of susceptible and infected is the same, then after 1 month at least* 50% *of the population will be in the recovered group.*

(Print out your graphs to help support your claims about these statements.)

(5) (a) Using the system (∗), show that $\dfrac{dS}{dI} = \dfrac{aSI}{bI - aSI}$.

(b) If $I(0) = I_0$ and $S(0) = S_0$, show that $I(t) = I_0 + (S_0 - S(t)) + \dfrac{b}{a} \ln\left(\dfrac{S(t)}{S_0}\right)$.