Section 8.2: Logic Networks

April 15, 2025

Abstract

We examine the relationship between the abstract structure of a Boolean algebra and the practical problem of creating (optimal!) logic networks for solving problems[1]. There is a fundamental equivalence between Truth Functions, Boolean Expressions, and Logic Networks which allows us to pass from one to the other. While a problem might be easiest formulated in terms of a truth function, we might then recast it as a Boolean expression. Then Boolean algebra provides us with a simple mechanism by which to simplify the expressions, and hence to simplify the underlying logic network, which we then feed into a logic network.

We'll examine the binary adder (and half-adder) as a particular example, which will later be implemented as Finite State Machines.

s_1	s_2	L
On	On	On
On	Off	Off
Off	On	Off
Off	Off	On

1 An Example Application, and Fundamental Parallels

Example: Two light switches, one light!

The problem is as follows: A light at the bottom of some stairs is controlled by two light switches, one at each end of the stairs. The two switches should be able to control the light **independently**. How do we wire the light?

s_1	s_2	$f(s_1, s_2)$
1	1	1
1	0	0
0	1	0
0	0	1

- A Truth Function: $f(s_1, s_2) = L$

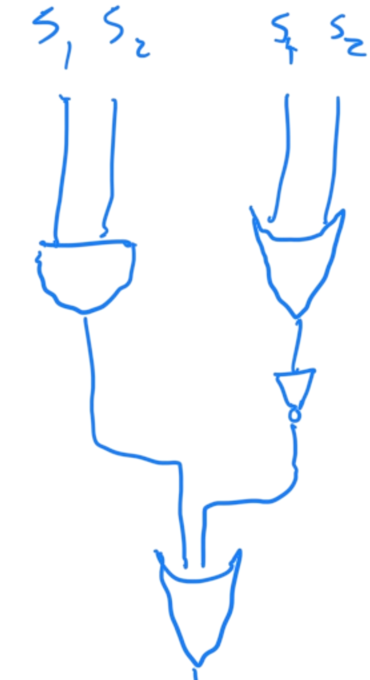
[1] From our text: "In 1938 the American mathematician Claude Shannon perceived the parallel between propositional logic and circuit logic and realized that Boolean algebra could play a part in systematizing this new realm of electronics."

$$(s_1 \wedge s_2) \vee (s_1' \wedge s_2')$$

$$\to \qquad s_1 \cdot s_2 + s_1' \cdot s_2' = f(s_1, s_2)$$

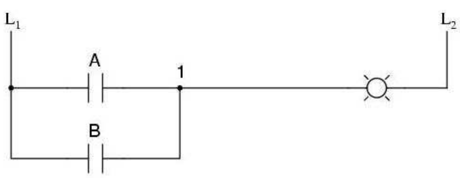
- A Boolean Expression (find two **equivalent Boolean expressions**)

$$f(s_1, s_2) = s_1 \cdot s_2 + (s_1 + s_2)'$$

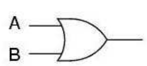


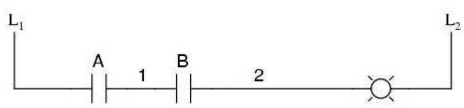
$f(s_1, s_2) = L$

- A Logic Network (Basic Components, Mechanics, and Conventions)
  - Input or output lines are not tied together except by passing through gates:
    * OR gate
    * AND gate

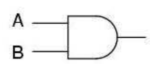


A	B	Output
0	0	0
0	1	1
1	0	1
1	1	1

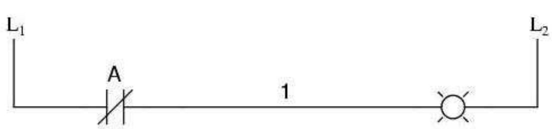




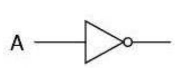
A	B	Output
0	0	0
0	1	0
1	0	0
1	1	1



    * NOT gate



A	Output
0	1
1	0



  - Lines can be split to serve as input to more than one device.
  - There are no loops, with output of a gate serving as input to the same gate. (feedback).

– There are no delay elements.

Figure 8.6, p. 638, shows how to wire an "or" – we do it in parallel ("and" is wired in series).

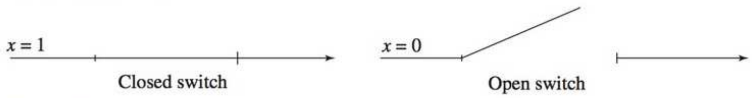


Figure 8.5

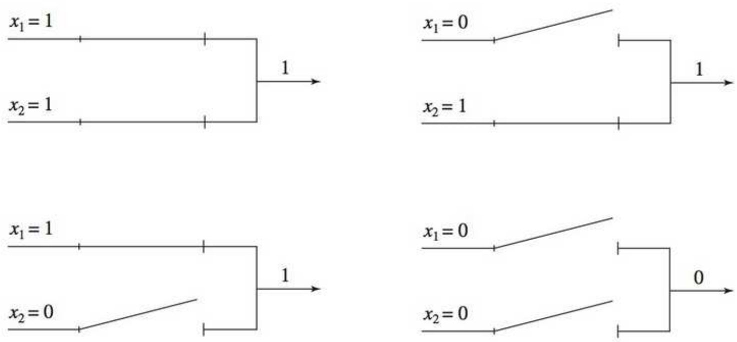


Figure 8.6

2 Applications

2.1 Converting Truth Tables to Boolean Expressions (Canonical Sum-of-Products Form)

Example: Practice 11, p. 645

PRACTICE 11

a. Find the canonical sum-of-products form for the truth function of Table 8.5.
b. Draw the network for the expression of part (a).

TABLE 8.5

x_1	x_2	x_3	$f(x_1, x_2, x_3)$	
1	1	1	1	*
1	1	0	0	
1	0	1	1	*
1	0	0	1	*
0	1	1	0	
0	1	0	0	
0	0	1	1	*
0	0	0	1	*

$f(x_1, x_2, x_3) =$

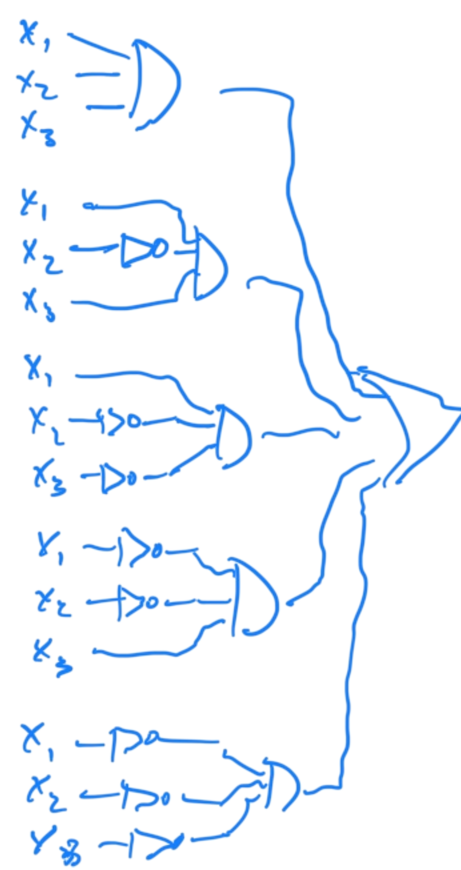
$x_1 x_2 x_3 +$

$x_1 x_2' x_3 +$

$x_1 x_2' x_3' +$

$x_1' x_2' x_3 +$

$x_1' x_2' x_3'$



Example: Exercise 15, p. 657 Find the canonical sum-of-products form for the truth function:

15.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

$f(x_1, x_2, x_3) =$

$x_1 x_2' x_3 +$
$x_1 x_2' x_3' +$
$x_1' x_2 x_3' =$

$x_1 x_2'(x_3 + x_3') +$
$x_1' x_2 x_3'$

(notice that you can easily simplify that canonical sum-of-products, using some Boolean algebra.)

$= x_1 x_2' + x_1' x_2 x_3'$

2.2 Converting Boolean Expressions to Logic Networks

Example: Practice 11, p. 645 (reprise)

PRACTICE 11

a. Find the canonical sum-of-products form for the truth function of Table 8.5.
b. Draw the network for the expression of part (a).

TABLE 8.5

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

x_1	x_2	x_3	f
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

Example: Exercise 2, p. 655 Write a truth function and construct a logic network using AND gates, OR gates, and inverters for the Boolean expression $(x_1 + x_2') + x_1' x_3$

$f(x_1, x_2, x_3) = (x_1' x_2 x_3')'$

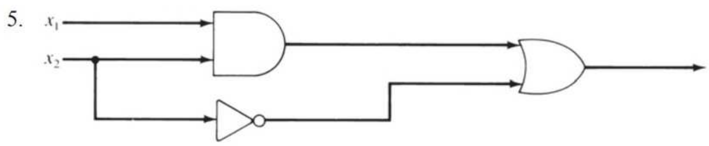
2.3 Converting Logic Networks to Truth Functions or Boolean Expressions

Example: Exercise 5, p. 655

x_1	x_2	$f(x_1, x_2)$
1	1	1
1	0	1
0	1	0
0	0	1

For Exercises 5–8, write a Boolean expression and a truth function for each of the logic networks shown.

5.



$$(x_1 \cdot x_2) + x_2' = f(x_1, x_2)$$

$$= (x_1' x_2)'$$

2.4 Simplifying Canonical Form

We can use properties of Boolean algebra to simplify the canonical form, creating a much simpler logic network as a result.

Example: Practice 11, p. 645 (reprise)

PRACTICE 11

a. Find the canonical sum-of-products form for the truth function of Table 8.5.
b. Draw the network for the expression of part (a).

TABLE 8.5

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

$x_1 x_3$

$x_1 \cdot x_2'$

$x_1' \cdot x_2'$

x_2'

$$f(x_1, x_2, x_3) = x_1 x_3 + x_2'$$

Wouldn't it be nice if there were some systematic way of doing this? That's the subject matter of the next section! We'll see two different ways to simplify a cannonical sum of products.

2.5 An example: Adding Binary numbers

2.5.1 Half-Adders

Half-Adder: Adds two binary digits.

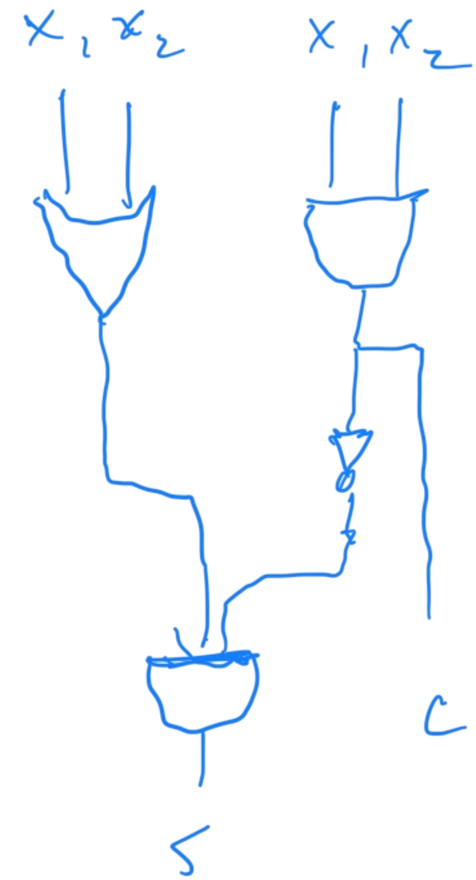
$$s = x_1' x_2 + x_1 x_2'$$
$$c = x_1 x_2$$

x_1	x_2	s	c
1	1	0	1
1	0	1	0
0	1	1	0
0	0	0	0

s is the result of an "XOR" operation (exclusive or) of the two inputs, whereas c is the product of the two inputs. Note, however, that the half-adder doesn't implement s in this way: instead,

$$s = (x_1 + x_2) \cdot (x_1 x_2)' =$$

$$(x_1 + x_2)(x_1' + x_2') =$$
$$x_1x_1' + x_1x_2' + x_2x_1' + x_2x_2' =$$
$$0 + x_1x_2' + x_2x_1' + 0 =$$
$$x_1x_2' + x_2x_1'$$



Questions:

a. How?

b. Why? $c = x_1 \cdot x_2$

$s = (x_1 + x_2) \cdot c'$

2.5.2 Full-Adders

Full-Adder: Adds two digits plus the carry digit from the preceding step (which we can create out of two half-adders!).

- Given the preceding carry digit c_{i-1}, and binary digits x_i and y_i.
- We'll use a half-adder to add x_i to y_i, obtaining write digit σ and carry digit γ.
- Then use a half-adder to add the carry digit c_{i-1} to σ; the write digit is s_i, and call the carry digit c.
- To get the carry digit c_i, compare the carry digits c and γ: if either gives a 1, then $c_i = 1$ (so it's an "or").

Let's derive all that from the truth functions, representing the sum from the full-adder:

c_{i-1}	x_i	y_i	c_i	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

So the canonical sum of products forms of each function are

$$\begin{aligned} s_i(c_{i-1}, x_i, y_i) &= c'_{i-1}x'_iy_i \\ &+ c'_{i-1}x_iy'_i \\ &+ c_{i-1}x'_iy'_i \\ &+ c_{i-1}x_iy_i \\ &= c'_{i-1}\underbrace{(x'_iy_i + x_iy'_i)}_{\sigma} + c_{i-1}\underbrace{(x'_iy_i + x_iy'_i)'}_{} \end{aligned}$$

write digit from the sum of σ & c_{i-1}

write digit from $x_i + y_i$

write digit negated

and

$$
\begin{aligned}
c_i(c_{i-1}, x_i, y_i) \quad = \quad & c'_{i-1}x_iy_i \\
+ \quad & c_{i-1}x'_iy_i \\
+ \quad & c_{i-1}x_iy'_i \\
+ \quad & c_{i-1}x_iy_i \\
= \quad & x_iy_i + c_{i-1}(x'_iy_i + x_iy'_i)
\end{aligned}
$$

carry digit from the half adder of x_i, y_i

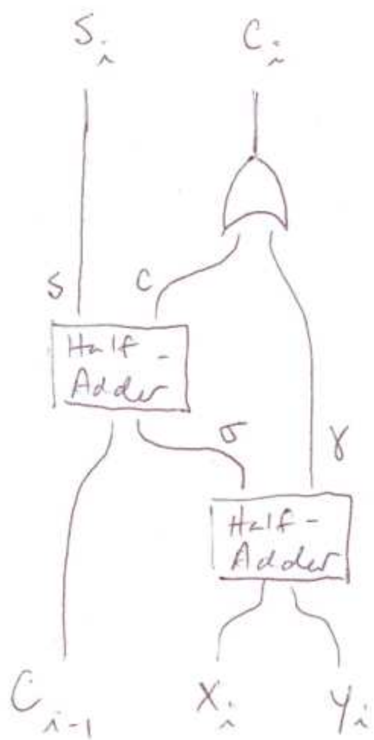
γ

σ

We recognize these quantities in terms of half-adders:

$c = c_{i-1} \cdot \sigma$

- We recognize the write digit $\sigma = x'_iy_i + x_iy'_i$ and the carry digit $\gamma = x_iy_i$ of the half-adder of x_i and y_i.
- Then s_i is just the write digit s of the half-adder of c_{i-1} and σ;
- Meanwhile, c_i is the sum of γ and the carry digit c of the half-adder of c_{i-1} and σ.
- That is illustrated in this sad figure I once drew:



start w/ 0 carry digit

c_{i-1}	x_i	y_i	σ	γ	s_i	c_i
0	1	1	0	1	1	1
1	0	1	1	0	0	1
1	1	1	0	1	1	1
1	0	0	0	0	1	0

Figure 1: The full-adder takes input digits x_i and y_i, as well as the carry digit c_{i-1} from the previous step and computes write digit s_i and carry digit c_i. Then do it again!

Example: Practice 12, p. 650 Trace the operation of the circuit as it adds 101 and 111.

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 1110
 0101
 0111
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 1100
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