$$S(1) = a$$
$$S(n) = c\,S(n-1) + g(n)$$

This is the recurrence relation

To prove:

$$S(n) = c^{n-1} \cdot S(1) + \sum_{i=2}^{n} c^{n-i} g(i)$$

(the closed form solution for $S(n)$)

Expand:

$$S(2) = c \cdot S(1) + g(2)$$
$$= c \cdot a + g(2)$$
$$S(3) = c \cdot S(2) + g(3)$$
$$= c\left(c \cdot a + g(2)\right) + g(3)$$
$$= c^2 a + c\,g(2) + g(3)$$
$$S(4) = c\,S(3) + g(4)$$
$$= c\left(c^2 a + c\,g(2) + g(3)\right) + g(4)$$
$$= c^3 a + c^2 g(2) + c\,g(3) + g(4)$$

Guess

$$S(5) = c^4 a + c^3 g(2) + c^2 g(3) + c\,g(4) + g(5)$$

Guess more general:

$$S(n) = c^{n-1} a + \sum_{i=2}^{n} c^{n-i} g(i)$$

Verify (by induction)

$$S(2) = c^{2-1}a + \sum_{i=2}^{2} c^{2-i} g(i)$$

$$= ca + g(2) \checkmark$$

Assume $P(k)$: $S(k) = c^{k-1}a + \sum_{i=2}^{k} c^{k-i} \cdot g(i)$

Consider $P(k+1)$: $S(k+1) = c^{(k+1)-1}a + \sum_{i=2}^{k+1} c^{k+1-i} g(i)$

& its LHS:

$$S(k+1) = cS(k) + g(k+1) \quad \text{(from the RR)}$$

$$= c\left[c^{k-1}a + \sum_{i=2}^{k} c^{k-i} g(i)\right] + g(k+1)$$

$$= c^{(k+1)-1}a + \sum_{i=2}^{k} c^{k+1-i} g(i) + g(k+1)$$

$1 = c^0$

$$= c^{(k+1)-1}a + \sum_{i=2}^{k} c^{(k+1)-i} g(i) + \underbrace{c^{(k+1)-(k+1)}}_{=1=c^0} g(k+1)$$

$$= c^{(k+1)-1}a + \underbrace{\sum_{i=2}^{k+1} c^{(k+1)-i} g(i)}_{\text{incorporate } i=k+1 \text{ term}} \checkmark$$

But this makes it look like the $i=k+1$ term.

$\therefore$ by the 1st principle of induction

True for all n.

$$T(0) = 1 \qquad c = 1$$

$$T(n) = n + 1 \qquad n = 1$$

$$g(n) = 1$$

$$T(n) = 1 \cdot T(1) + \sum_{i=2}^{n} 1^{n-i} \cdot 1$$

$$= 1 \cdot 2 + \sum_{i=2}^{n} 1$$

$$= 2 + n - 1$$

$$= n + 1$$