Example: **Practice 7 (or "Gauss's theorem"), p. 115** Prove that, for any natural number n, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Example: **Exercise 39, p. 125** Prove that $n! \geq 2^{n-1}$ for $n \geq 1$.

$P(n)$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Assertion: $(\forall n)\underbrace{\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right]}_{P(n)}$

Base Case: $P(1)$: $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$

$$\sum_{i=1}^{1} i = 1 = \frac{1 \cdot 2}{2} \quad \checkmark$$

Inductive Step: $P(m) \to P(m+1)$

Assume $P(m)$: $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$

Consider $P(m+1)$: $\sum_{i=1}^{m+1} i = \frac{(m+1)((m+1)+1)}{2}$

(in particular, the left-hand side):

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + (m+1) = \underbrace{\frac{m(m+1)}{2}}_{\text{by assumption of } P(m)} + m+1$$

$$= \frac{m(m+1)}{2} + \frac{(m+1)\cdot 2}{2} = \frac{m(m+1) + 2(m+1)}{2}$$

$$= \frac{(m+1)(m+2)}{2} = \frac{(m+1)((m+1)+1)}{2} \quad \checkmark$$

$P(m+1)$ is true.

$\therefore$ $(\forall n)\, P(n)$ by the 1^{st} principle of mathematical induction.

$P(n)$: $n! \geq 2^{n-1}$ for $n \geq 1$

Theorem : $(\forall n)\, P(n)$. Proof by induction:

Base Case : $P(1)$: $1! \geq 2^{1-1}$

$$1! = 1 = 2^0 = 2^{1-1} \checkmark$$

Inductive Step : $P(m) \rightarrow P(m+1)$

Assume $P(m)$: $m! \geq 2^{m-1}$ $\quad m \geq 1$

Consider $P(m+1)$: $(m+1)! \geq 2^{(m+1)-1}$

in particular the LHS :

$$(m+1)! = m! \cdot (m+1) \geq \underbrace{2^{m-1}}_{\text{By assumption}} \cdot (m+1)$$

We know $m \geq 1$, so $m+1 \geq 2$

$$\geq 2^{m-1} \cdot 2 = 2^{(m+1)-1}$$ ✓

$\therefore P(m+1)$

$\therefore (\forall n)\, P(n)$ by mathematical induction.

Ex. 48: Prove that $3^{2n}+7$ is divisible by 8.

Defn of $3^{2n}+7$ is divisible by 8:

$$(\exists k)\left[\, 3^{2n}+7 = 8k \,\right]$$

$$P(n): \ (\exists k)\left[\, 3^{2n}+7 = 8k \,\right]$$

<u>Base Case:</u> $P(1): (\exists k)\left[\, 3^{2}+7 = 8k \,\right]$

$$3^2+7 = 9+7 = 16 = 8 \cdot 2$$

$$\therefore (\exists k)\left[\, 3^{2}+7 = 8k \,\right] \ ✓$$

<u>Inductive Step:</u> Assume $P(m)$ & show $P(m+1)$.

1. $P(m)$ hyp

2. $(\exists k)[3^{2m}+7=8k]$ (defn)

3. $3^{2m}+7=8k$ 2, ei

4. $3^2[3^{2m}+7]=3^2\cdot 8\cdot k$ number facts

5. $3^{2(m+1)}+63=(3^2k)8$

6. $3^{2(m+1)}+7+56=(3^2k)8$

7. $3^{2(m+1)}+7=(3^2k)8-56$

$$=(3^2k)\cdot 8-7\cdot 8$$

$$=\underbrace{[3^2k-7]}_{\text{integer}}\cdot 8$$

8. $3^{2(m+1)}+7=8j$ where j is an integer

9. $(\exists k)[3^{2(m+1)}+7=8k]$ 8, e.g.

10. $P(m+1)$

✓

$\therefore (\forall n)P(n)$ by mathematical induction.

Prove that $P(n): (x^n)' = nx^{n-1}$ for all natural numbers.

(we assume the product rule as a lemma).

Base Case: $P(1)$; $(x^1)' = 1 \cdot x^{1-1} = 1 \cdot 1 = 1$

$$\underbrace{(x)' = 1}_{\text{(proven w/ limit defn.)}} = 1 \cdot x^0 = 1 \cdot x^{1-1}$$

Inductive step: Assume $P(n)$. Consider the LHS of $P(m+1)$:

$$(x^{m+1})' = (x \cdot x^m)' = (x)' x^m + (x)(x^m)'$$

$$= 1 \cdot x^m + x \cdot m x^{m-1}$$

$$= 1 \cdot x^m + m x^{m-1+1}$$

$$= (1+m) x^m$$

$$= (1+m) x^{(1+m)-1}$$

$\therefore (\forall n) P(n)$ by the the 1^{st} principle of mathematical induction.

	$\boxed{23}$		13
X	3	X	2
	20		$11 = 8 + \boxed{13}$
A	2	A	3
	$18 = 13 + 5$		8 ✓
X	4	X	1
	$14 = 13 + 1$		7
A	1	A	2
	13 ✓		5 ✓
		X	1
			4
		A	1
			3 ✓

1
1
2
3
5
8
13
21

$$23 = 21 + 2$$

$$26 = 21 + 5$$