

Section 1.3: Quantifiers, Predicates, and Validity

January 19, 2023

Abstract

We now inject logical **variables** into the mix, and investigate wffs which describe properties of the domains of those variables in given “interpretations.” We still test their truth values, either for the specific domain in question, or even in all domains (validity).

1 Predicates and quantifiers

- **quantifier**: describes **how many** objects in a given domain have a certain property.

Examples:

- **universal quantifier** - \forall - “for any”, “for every”, “for all”
- **existential quantifier** - \exists - “there exists”, “for at least one”, “for some”

Have you encountered these quantifiers before, in other courses?

Lewis Carroll said that there are **three** types of propositions:

- Some cakes are nice.
- No cakes are nice.
- All cakes are nice.

(Why do we need only **two** quantifiers? Because we have **negation!**)

- **predicate**: a property of a variable (e.g. “ x is prime”), generally containing one or more variables (and perhaps some constants).

We combine the quantifiers and predicates to create expressions (predicate wffs) such as

$$(\forall x)P(x)$$

$$(\forall y)P(y)$$

False in the domain of integers, where

Universe of cakes
 $(\exists x) N(x)$ - some cakes are nice
 $(\forall x) N(x)$ - all cakes are nice
 $(\forall x) N(x)'$ - all cakes are not nice

$P(x) - x$ is prime.

which we then must interpret. For example, this might be said in the context of the integers, with $P(x)$ standing for “ x is prime”. (So this wff would be false in this context. The same wff would be true – but trivially – in the context of all prime numbers!)

There is nothing special about the variable x , so this wff is the same as $(\forall y)P(y)$, $(\forall z)P(z)$, etc. We say that x is a dummy variable.

Predicates may have any number of variables in them: the example above is a unary predicate, with only a single variable.

• Truth value hence now depends on the **Interpretation** of an expression:

- **domain of interpretation** – a non-empty set to which the predicate expression is applied;
- assignment of a property of the objects to each predicate in the expression;
- assignment of particular objects to each constant symbol in the expression.

NB: Universe is non-empty
(be kind of silly to do logic on an empty universe!)

We start with something abstract, and replace it with concrete instances in a given context.

Example: Selections from #3, p. 50: What is the truth value of each in the interpretation where the domain is the integers?

- a. $(\forall x) [(\exists y)(x + y = x)]$ T
- b. $(\exists y) [(\forall x)(x + y = x)]$ F
- c. $(\forall x) [(\exists y)(x + y = 0)]$ T
- d. $(\exists y) [(\forall x)(x + y = 0)]$ F
- e. $(\forall x) [(\forall y)(x < y \vee y < x)]$ F (x=y)
- f. $(\forall x) [x < 0 \rightarrow (\exists y)(y > 0 \wedge x + y = 0)]$ T

sometimes you can switch the order of quantifiers, sometimes you can't!

Example: #7(a, c), p. 51. For each wff, find an interpretation in which it is true, and one in which it is false.

- a. $(\forall x)([A(x) \vee B(x)] \wedge [A(x) \wedge B(x)]')$
- c. $(\forall x)[P(x) \rightarrow (\exists y)Q(x, y)]$

universe - people
 $P(x) - x$ is a person
 $Q(x, y) - y$ is the mother of x

In exercise 7(c), the predicate $Q(x, y)$ is an example of a **binary** predicate.

True: even or odd (but not both)
 False: positive or negative (0!)

True
 False if we replace $Q(x, y)$ w/ y is the queen of x .

exclusive of: Integers

- **scope** of a quantifier: the portion of a predicate to which the quantifier applies; indicated with parentheses or brackets (but these may be neglected **if the scope is clear**).

- **free variable**: a variable in a predicate wff outside the scope of a quantifier involving that variable.

Example: Ex. 9(c), p. 51. Indicate the scope of each quantifier in $(\exists x)[(\forall y)P(x, y) \wedge Q(x, y)]$.

y is free
The scope of $(\forall y)$ is just $P(x, y)$
scope of $(\exists x)$

2 Translation of English statements into predicate wffs

2.1 “there exists” with “and” – \exists with \wedge “for all” with “implies” – \forall with \rightarrow

As noted before, translation can be a very tricky business, but it’s obviously an important one. As is often the case, the process of translation does not result in a unique expression: there may be several different ways to “say the same thing” in wffs.

Our author encourages us to remember that

- typically \exists and \wedge go together, whereas
- typically \forall and \rightarrow go together.

I can’t emphasize it enough! Also, a single English “formula” or theorem may be given by numerous wffs: the **redundancy in our connectives ensures that**.

Example: Ex. 21(c, e, g), p. 54. Write each statement as a predicate wff:

- c. Some lawyers admire only judges.

Domain people
 $(\exists x)[L(x) \wedge (\forall y)[A(x, y) \rightarrow J(y)]]$

- e. Only judges admire judges.

$(\forall x)(\forall y)[A(x, y) \wedge J(y) \rightarrow J(x)]$

- g. Some women admire no lawyer.

$(\exists x)[W(x) \wedge (\forall y)(A(x, y) \rightarrow \neg L(y))]$

*$W(x)$ – x is a woman
 $A(x, y)$ – x admires y*

Example: Ex. 23(a-d), p. 54. Write each as a predicate wff:

- a. All bees love all flowers.

$(\forall x)(\forall y)[B(x) \wedge F(y) \rightarrow L(x, y)]$ *$(L(y) \rightarrow A(x, y))$*

- b. Some bees love all flowers.

$(\exists x)[B(x) \wedge (\forall y)[F(y) \rightarrow L(x, y)]]$

- c. All bees love some flowers.

$(\forall x)(\exists y)[B(x) \wedge F(y) \rightarrow L(x, y)]$

- d. Every bee hates only flowers.

$L(x, y)$ – x loves y

Universe – living creatures

\exists would work for the contrapositive

2.2 Negation

$$(\forall x)[B(x) \rightarrow (\exists y)[E(y) \wedge L(x,y)]]$$

Negation of predicate wffs: some cases are standard, e.g.

- The negation of "Every x has property A." is "There is an x which doesn't have property A."; or "There is an x which has property A'."

$$[(\forall x)A(x)]' \iff (\exists x)[A(x)']$$

$$\begin{aligned} & \text{d,} \\ & (\forall x)[B(x) \rightarrow (\exists y)[H(x,y) \\ & \quad \rightarrow F(y)]] \end{aligned}$$

- The negation of "There is an x which has property A." is "No x has property A."; or "Every x has property A'."

$$[(\exists x)A(x)]' \iff (\forall x)[A(x)']$$

In general, English makes negation kind of tricky. Watch your step!

Example: Ex. 27(c,d), p. 56. Negate each:

$$\begin{aligned} & (\forall x)(Tall(x) \wedge Thin(x)) \xrightarrow{\text{negation}} \\ & (\exists x)(Tall(x)' \vee Thin(x)') \\ & \hline & (\forall x)[old(x)' \text{ or } faded(x)'] \end{aligned}$$

c. All people are tall and thin.

d. Some pictures are old and faded.

3 Validity

The truth value of a predicate wff depends on the interpretation, but there are some for which the wff is true independent of the interpretation. These are called **valid** predicate wffs (the analogue of tautology for propositional wffs).

Whereas we can check the "validity" of a propositional wff (just check the truth table to see if it's a tautology), there is no general check for the validity of a predicate wff, since it depends on context. In spite of that, there are some valid predicate wffs (context free truth!), as demonstrated in the text:

$$\begin{aligned} & (\forall x)P(x) \rightarrow (\exists x)P(x) \\ & (\forall x)P(x) \rightarrow P(a) \\ & P(x) \rightarrow (Q(x) \rightarrow P(x)) \end{aligned}$$

$$\iff [P(x) \wedge Q(x) \rightarrow P(x)]$$

Example: Ex. 33(d,e), p. 57. Explain why each wff is valid:

d. $A(a) \rightarrow (\exists x)A(x)$

If a has property A, then there is some x in the domain that has property A.

e. $(\forall x)[A(x) \rightarrow B(x)] \rightarrow [(\forall x)A(x) \rightarrow (\forall x)B(x)]$

If for every x in the domain $A(x) \rightarrow B(x)$ (having property A means you have property B) then if every x has property A, then every x has property B.