Section 1.3: Quantifiers, Predicates, and Validity

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Abstract

We now inject logical **variables** into the mix, and investigate wffs which describe properties of the domains of those variables in given "interpretations." We still test their truth values, either for the specific domain in question, or even in all domains (validity).

1 Predicates and quantifiers

- **quantifier**: describes **how many** objects in a given domain have a certain property.

  Examples:

  - **universal quantifier** - $\forall$ - "for any", "for every", "for all"
  - **existential quantifier** - $\exists$ - "there exists", "for at least one", "for some"

  Have you encountered these quantifiers before, in other courses? Lewis Carroll said that there are **three** types of propositions:

  a. Some cakes are nice. $(\exists x)[C(x) \wedge N(x)]$
  b. No cakes are nice. $(\forall x)[C(x) \to N(x)']$
  c. All cakes are nice. $(\forall x)[C(x) \to N(x)]$

  Why do we need only **two** quantifiers? Because we have **negation**!: "No cakes are nice" means "all cakes are **not** nice!"

- **predicate**: a property of a variable (e.g. "x is prime"), generally containing one or more variables (and perhaps some constants).

  We combine the quantifiers and predicates to create expressions (predicate wffs) such as

$$(\forall x)P(x)$$

which we then must *interpret*. For example, this might be said in the context of the integers, with $P(x)$ standing for "x is prime". (So this wff would be false in this context. The same wff would be true – but trivially – in the context of all prime numbers!)

There is nothing special about the variable x, so this wff is the same as $(\forall y)P(y)$, $(\forall z)P(z)$, etc. We say that x is a *dummy* variable.

Predicates may have any number of variables in them: the example above is a *unary* predicate, with only a single variable.

- Truth value hence now depends on the **Interpretation** of an expression:
  - **domain of interpretation** – a **non-empty** set to which the predicate expression is applied;
  - assignment of a property of the objects to each predicate in the expression;
  - assignment of particular objects to each constant symbol in the expression.

We start with something abstract, and replace it with concrete instances in a given context.

Example: Selections from #3, p. 50: What is the truth value of each in the interpretation where the domain is the integers?

a. $(\forall x)\,[(\exists y)(x + y = x)]$ T $(y = 0)$

b. $(\exists y)\,[(\forall x)(x + y = x)]$ T $(y = 0)$

c. $(\forall x)\,[(\exists y)(x + y = 0)]$ additive inverses ✓ T

d. $(\exists y)\,[(\forall x)(x + y = 0)]$ F no magic add. inverse!

e. $(\forall x)\,[(\forall y)(x < y \vee y < x)]$ F $(x = y$ is a counter example$)$

f. $(\forall x)\,[x < 0 \to (\exists y)(y > 0 \wedge x + y = 0)]$ T add inverses exist for all negative integers.

Example: #7(a, c), p. 51. For each wff, find an interpretation in which it is true, and one in which it is false.

Universe: NKU classes 9am MWF

Exclusive Or

a. $(\forall x)([A(x) \vee B(x)] \wedge [A(x) \wedge B(x)]')$

T: $A(x)$: x is a course in MP.

$B(x)$: $A(x)'$ – x is not a class in MP.

F: $B(x)$: x is a course in Science building.

c. $(\forall x)[P(x) \to (\exists y)Q(x, y)]$

In exercise 7(c), the predicate $Q(x, y)$ is an example of a **binary** predicate.

True:

Universe – all people.

$P(x)$ - x is a parent

$Q(x,y)$ ~ y is the child of x

For all people, if x is a parent then there is a person y who is their child.

- **scope** of a quantifier: the portion of a predicate to which the quantifier applies; indicated with parentheses or brackets (but these may be neglected **if the scope is clear**).

False: $Q(x,y)$ - y is the sister of x.

- **free variable**: a variable in a predicate wff outside the scope of a quantifier involving that variable.

Example: Ex. 9(c), p. 51. Indicate the scope of each quantifier in $(\exists x)[(\forall y)P(x,y) \wedge Q(x,y)]$.

scope of $(\forall y)$

y is free here, not quantified by $(\exists x)$ or $(\forall y)$.

2 Translation of English statements into predicate wffs

2.1 "there exists" with "and" – ∃ with ∧ "for all" with "implies" – ∀ with →

As noted before, translation can be a very tricky business, but it's obviously an important one. As is often the case, the process of translation does not result in a unique expression: there are always several different ways to "say the same thing" in wffs.

Our author encourages us to remember that

- typically ∃ and ∧ go together, whereas
- typically ∀ and → go together.

I can't emphasize it enough! Also, a single English "formula" or theorem may be given by numerous wffs: the **redundancy in our connectives ensures that**.

Example: Ex. 21(c, e, g), p. 54. Write each statement as a predicate wff: $A(x,y)$ - x admires y

Universe is all people.

c. Some lawyers admire only judges. $(\exists x)\{L(x) \wedge (\forall y)[A(x,y) \to J(y)]\}$

e. Only judges admire judges. $(\forall x)(\forall y)[(J(x) \wedge A(y,x)) \to J(y)]$

g. Some women admire no lawyer. $(\exists x)[W(x) \wedge (\forall y)[A(x,y) \to L(y)']]$

Example: Ex. 23(a-d), p. 54. Write each as a predicate wff: Universe is all living things

a. All bees love all flowers. $(\forall x)(\forall y)[B(x) \wedge F(y) \to L(x,y)]$

b. Some bees love all flowers. $(\exists x)(B(x) \wedge (\forall y)[F(y) \to L(x,y)])$

c. All bees love some flowers. $(\forall x)\{B(x) \to (\exists y)[F(y) \wedge L(x,y)]\}$

d. Every bee hates only flowers. $(\forall x)(\forall y)[B(x) \wedge L(x,y)' \to F(y)]$

2.2 Negation

Good news: this is very straightforward:

Negation of predicate wffs: some cases are standard, e.g.

- The negation of "Every x has property A." is "There is an x which doesn't have property A."; or "There is an x which has property A'."

$$[(\forall x)A(x)]' \iff (\exists x)[A(x)']$$

symmetric (or "dual")

- The negation of "There is an x which has property A." is "No x has property A."; or "Every x has property A'."

$$[(\exists x)A(x)]' \iff (\forall x)[A(x)']$$

In general, English makes negation kind of tricky. Watch your step!

Example: Ex. 27(c,d), p. 56. Negate each:

c. All people are tall and thin. $[(\forall x)[P(x) \to Tall(x) \wedge Thin(x)]]'$

$(\exists x)[P(x) \to (Tall(x) \wedge Thin(x))]'$

$(\exists x)[P(x) \wedge (Tall(x) \wedge Thin(x))']$

$(\exists x)[P(x) \wedge (Tall(x)' \vee Thin(x)')]$

$\Updownarrow$ "for all" becomes "there exists" w/negation

d. Some pictures are old and faded. $[(\exists x)[P(x) \wedge O(x) \wedge F(x)]]'$

$(\forall x)[P(x) \wedge (O(x) \wedge F(x))]' \iff$ (de Morgan)

$(\forall x)[P(x)' \vee (O(x) \wedge F(x))']$

$\iff$ (implication)

$(\forall x)(P(x) \to (O'(x) \vee F'(x)))$

associativity says we can add parentheses

3 Validity

The truth value of a predicate wff depends on the interpretation, but there are some for which the wff is true independent of the interpretation. These are called **valid** predicate wffs (the analogue of tautology for propositional wffs).

Whereas we can check the "validity" of a propositional wff (just check the truth table to see if it's a tautology), there is no general check for the validity of a predicate wff, since it depends on context. In spite of that, there are some valid predicate wffs (context free truth!), as demonstrated in the text:

$$(\forall x)P(x) \to (\exists x)P(x)$$
$$(\forall x)P(x) \to P(a)$$
$$P(x) \to (Q(x) \to P(x)) \iff \underbrace{P(x) \wedge Q(x) \to P(x)}$$

deduction method

This is a clearer wff for us to interpret - I hope you agree!

Example: Ex. 33(d,e), p. 57. Explain why each wff is valid (say it in words):

d. $A(a) \to (\exists x)A(x)$

e. $(\forall x)[A(x) \to B(x)] \to [(\forall x)A(x) \to (\forall x)B(x)]$