Section 8.1: Boolean Algebra Structure

April 20, 2022

Abstract

First of all, note that we're only reading 8.1 through p. 626 (up to Isomorphic Boolean Algebras).

A Boolean algebra (named after George Boole) is a generalization of both the propositional logic and the set theory we studied earlier this term. We are going to focus on using it to understand the basic elements of (computer) logic[1], however, which is based on a binary (0,1) alphabet.

In this first section we are introduced to the fundamental concepts of Boolean algebra.

1 Definition and Terminology

Definition: a **Boolean Algebra** is a set B on which are defined two binary operations $+$ and $\cdot$, and one unary operation $'$, and in which there are two distinct elements 0 and 1 such that the following properties hold for all $x, y, z \in B$:

$1a.\ x + y = y + x$	$1b.\ x \cdot y = y \cdot x$	commutative property
$2a.\ (x + y) + z = x + (y + z)$	$2b.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$	associative property
$3a.\ x + (y \cdot z) = (x + y) \cdot (x + z)$	$3b.\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$	distributive property
$4a.\ x + 0 = x$	$4b.\ x \cdot 1 = x$	identity property
$5a.\ x + x' = 1$	$5b.\ x \cdot x' = 0$	complement property

The element x' is called the **complement** of x. The algebra may be denoted $[B, +, \cdot, ', 0, 1]$.

Of these properties, certainly the distributive property 3a. may seem the strangest, since it obviously doesn't hold for the usual suspects $+$ and $\cdot$. However these aren't the usual suspects!

[1] "I am now about to set seriously to work upon preparing for the press an account of my theory of Logic and Probabilities which in its present state I look upon as the most valuable if not the only valuable contribution that I have made or am likely to make to Science and the thing by which I would desire if at all to be remembered hereafter...." George Boole. "Boole's work has to be seen as a fundamental step in today's computer revolution." (from his bio)

Notice the beautiful **symmetry** (or **duality**) in this definition: the roles of $+$ and $\cdot$ are exactly reversed, as are the special elements 0 and 1.

Question: how are these reflected in the properties of propositional logic that we studied earlier this term? In set theory?

A change in notation: Speaking of propositional logic, as we move forward one change that makes sense is to switch to thinking of truth *functions*, instead of wffs:

$$f : \{T, F\}^n \rightarrow \{T, F\}$$

The function f take elements of the Cartisian product $\{T, F\}^n$ into the set $\{T, F\}$. We're doing algebra, after all, so it seems reasonable that we'll want to operate on variables with functions.

So we'll want to think of implication, for example, as a function of two variables (wffs) of the form $f : \{T, F\}^2 \rightarrow \{T, F\}$. If we wrote out the truth table, there would be four rows for the domain (all ordered pairs of T, F), and the range values would be in the right column.

A	B	$A \longrightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Furthermore, we'll want to replace "T" and "F" with 1 and 0 from here on out.

$x \cdot 1 = x$

There's an advantage to the function notation: we can speak of two functions being equal (=), to mean that their corresponding wffs are equivalent ($\iff$). Equality is a little easier to throw around....

$0 \cdot 1 = 0$ ✓

$1 \cdot 1 = 1$ ✓

In Example 2, p. 621, which illustrates the world's simplest Boolean Algebra, the set $B = \{0, 1\}$ consists of **only** two elements (so they must be our distinguished elements), and the binary operations of $+$ and $\cdot$ are given by $x + y = max(x, y)$ and by $x \cdot y = min(x, y)$. Complements are given by $0' = 1$ and $1' = 0$.

Example: Practice 1, p. 621 : Verify property 4b for the Boolean algebra of Example 2.

1.1 Idempotence

Curiously enough, $x + x = x$ in a Boolean algebra (this is the **idempotent property**. You'll want to remember that one, for any proofs!) And since $x + x = x$, we must have $x \cdot x = x$ by the beautiful symmetry of the operations. This symmetry, which you have already encountered as **duality**, means that we only have to do half the work most of the time (or that we oftentimes get something for free!).

You may have bumped into the notion of idempotence in linear algebra: for example, projection matrices are idempotent, such as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x \cdot x = x \cdot x + 0$ 4a
$= x \cdot x + x \cdot x'$ 5b
$= x \cdot (x + x')$ 3b
$= x \cdot 1$ 5a
$= x$ 4b

This matrix projects three-dimensional space onto the first and second dimensions; and projecting onto those dimensions a second time doesn't change anything (i.e., $A \cdot A = A$). (What's the shadow of a shadow?) However, in a Boolean algebra, this property is true **for every element**!

Example: Practice 2, p. 624

a. What does the idempotent property become in the context of propositional logic?

b. What does it become in the context of set theory?

c. Let's prove the dual property $x \cdot x = x$. We get it for free by duality (but we can prove it, of course, using one of my favorite tricks in the book).

Example: Practice 3, p. 624 (universal bound property)

a. Prove that the property $\boxed{x + 1 = 1}$ holds in any Boolean algebra. Give a reason for each step.

b. What is the dual property?

$\boxed{x \cdot 0 = 0}$

$x + 1 = x + (x + x')$ 5a
$= (x + x) + x'$ assoc.
$= x + x'$ idempotence
$= 1$ complements

1.2 Complements are Unique

Given an element x of the set B of a Boolean algebra, the complement x' is the **unique** element of B with the property that

$$x + x' = 1 \text{ and } x \cdot x' = 0$$

Furthermore, if you ever find an element a such that

$$x + a = 1 \text{ and } x \cdot a = 0$$

then $a = x'$.

The proof is on p. 625. The author summarizes this by saying "if it walks like a duck, and it quacks like a duck, it must be a duck."

2 Hints for proving Boolean Algebra Equalities (p. 624)

- Usually the best approach is to start with the more complicated expression and try to show that it reduces to the simpler expression via the axioms of the Boolean algebra.
- It may help to frame the argument in terms of either set theory or propositional logic, to give you a framework for understanding the thrust of the argument.
- Think of adding some form of 0 (like $x \cdot x'$) or multiplying by some form of 1 (like $x + x'$). [These are my among my favorite tricks in mathematics!]
- Don't forget property 3a, the distributive property of addition over multiplication (just because it seems weird).
- Remember the idempotent properties: $x + x = x$ and $x \cdot x = x$.
- Remember the uniqueness of complements.

3 Examples

Example: Exercise 8, p. 633 : Prove De Morgan's Laws for any Boolean algebra, e.g.

$$\boxed{(x + y)' = x' \cdot y'}$$

By uniqueness of complements

Consider

$x+y + x'\cdot y' =$

$= y + x + x'\cdot y'$ commutativity

$= y + (x+x')\cdot(x+y')$ weird distributivity

$= y + 1\cdot(x+y')$ complements

$= y + x + y'$ identity

$= y + y' + x$ commutativity

$= 1 + x$ complements

$= 1$

Consider

$(x+y)\cdot x'\cdot y'$

$= (x\cdot x' + y\cdot x')\cdot y'$

$= (0 + y\cdot x')\cdot y'$

$= (y\cdot x')\cdot y'$

$= (x'\cdot y)\cdot y'$

$= x'\cdot(y\cdot y')$

$= x'\cdot 0$

$= 0$

Example: Exercise 14, p. 634 : Prove that in any Boolean algebra

$$x \cdot y' = 0 \iff x \cdot y = x$$

$\Rightarrow$

Assume $x \cdot y' = 0$

Consider

$$x \cdot y = x \cdot y + 0$$
$$= x \cdot y + x \cdot y'$$
$$= x \cdot (y + y')$$
$$= x \cdot 1$$
$$= x \checkmark$$

$\Leftarrow$

Assume $x \cdot y = x$

Consider

$$x \cdot y' = (x \cdot y) \cdot y'$$
$$= x \cdot (y \cdot y')$$
$$= x \cdot 0$$
$$= 0 \checkmark$$

Example: Exercise 16a, p. 634 : Prove that in any Boolean algebra

$$x + y = 0 \rightarrow (x = 0 \wedge y = 0)$$

Assume $x + y = 0$

WLOG Consider x (symmetric in y & x)

$x = x + 0$; identity

$= x + (x + y)$ assumption

$= (x + x) + y$ assoc.

$= x + y$ idempotence

$= 0$ ✓ assumption

(The proof for $y = 0$ is completely symmetric.)

4 One last cool thing...

This section ends "in the weeds" a bit, but the upshot is really interesting: the characterization of all finite Boolean algebras. It turns out that every **finite** Boolean algebra is of order (size) 2^n. **Can you think of a finite Boolean algebra with 2^n elements?**[2]

Then it turns out that every finite Boolean algebra is isomorphic to every other finite Boolean algebra of the same order. The power... of the Power set!

Example 2, p. 621, reprise: illustrating the world's simplest Boolean Algebra. This algebra must be isormorphic to the Boolean algebra created by a set of one element, whose power set has two elements: the empty set and the set itself. As usual, set union and intersection ($\cup$ and $\cap$) serve as the operations of $\cdot$ and $+$.

The empty set will serve as "0", whereas the set itself serves as "1". Complements are given by 0'=1 and 1'=0, of course!

[2] Subsets of a set S of order n, with intersection and union, and special elements S and $\oslash$.