Well-Ordering: every non-empty set of natural numbers contains a smallest element.

1st principle

Proof (by induction, & contradiction).

Let T be a non-empty subset of natural numbers (i.e. it has at least one integer m).

Assume it has no least element.

$P(n)$: Every member of T is $> n$.

Base case: $P(1)$: Every member of $T \geq 1$.

Must be true, or 1 is the smallest natural number in T, a least element.

Induction step: Assume $P(k)$ & show $P(k+1)$.

$P(k)$: every member of $T > k$.

What if $P(k+1)$ were false?

$P(k+1)$: every member of $T > k+1$.

$\therefore$ $k+1$ has to be in T, so it has a least element.

But it doesn't (by assumption).

$\therefore$ $P(k+1)$ must be true.

$P(k) \to P(k+1)$ for all $k \geq 1$.

$\therefore$ by the 1st principle of induction,

$(\forall n)\, P(n)$ for all natural numbers;

even $P(m)$ is true.

But that's a contradiction,

because $m \in T$.

So $P(m)$ is false.

$\therefore$ Well-ordering is established.