1 Climbing some trees...

Problem 1: Here's a really interesting problem, that connects trees to recursion, and the Catalan numbers in particular: Given $n \in \mathbb{N}$, find $s(n)$ – the number of structurally unique binary search trees that store values 1 through n. Find the recursion relation for $s(n)$, and its first 8 values. (For convenience let's define $s(0) = 1$.) This figure shows that $s(3) = 5$:

```
1        3    3      2      1
 \      /    /      / \      \
  3    2    1      1   3      2
 /    /      \                 \
2    1        2                 3
```

Using the same quaint graphing technique, but emphasizing the symmetry, we see that there are five possible BSTs for $n = 3$:

```
1        1         2         3         3
 \        \       / \       /         /
  3        2     1   3     2         1
 /          \             /           \
2            3           1             2
```

Or maybe this representation is more suggestive (with the $n = 2$ and $n = 1$ cases thrown in for good measure):

```
1 1        2        3 3
 \ \      / \      / /
  3 2    1   3    2 1                  2   1
 /   \           /   \                /     \
2     3         1     2              1       2            1
```

Problem 2: To test blood from blood donors for coronavirus, small samples of blood from $n = 2^m$ ($m \in \mathbb{N}$) donors are pooled, and then **the pooled sample is tested**. If negative, great – all clear, after just one test; if positive, however, apply the strategy recursively on pooled samples of two halves, one after the other.

a. **Best case scenario**: Suppose we know that **only one person** is infected (**and the lab knows it, too**): how many Covid-19 tests will the lab do in order to identify the individual? (You might consider some simple cases.) How does that compare to just testing everyone?

b. **Worst case scenario**: Suppose we know that **all** are infected (**but the lab doesn't know it**). How many tests will the lab do in order to determine that? How does that compare to just testing everyone?

c. **Reality**: is generally somewhere between best and worst cases. At what prevalence (infection rate) will sequential testing of all donors require about the same number of tests as this divide-and-conquer strategy?