EXAMPLE 13 A standard 64-square checkerboard is arranged in 8 rows of 8 squares each. Adjacent squares are alternating colors of red and black. A set of 32 1×2 tiles, each covering 2 squares, will cover the board completely (4 tiles per row, 8 rows). Prove that if the squares at diagonally opposite corners of the checkerboard are removed, the remaining board cannot be covered with 31 tiles.

The hard way to prove this result is to try all possibilities with 31 tiles and see that they all fail. The clever observation is to note that opposing corners are the same color, so the checkerboard with the corners removed has two less squares of one color than of the other. Each tile covers one square of each color, so any set of tiles must cover an equal number of squares of each color and cannot cover the board with the corners removed. ●

Common Definitions

Many of the examples in this section and many of the exercises that follow involve elementary **number theory**, that is, results about integers. It's useful to work in number theory when first starting to construct proofs because many properties of integers, such as what it means to be an even number, are already familiar. The following definitions may be helpful in working some of these exercises.

- A **perfect square** is an integer n such that $n = k^2$ for some integer k.
- A **prime number** is an integer $n > 1$ such that n is not divisible by any integers other than 1 and n.
- A **composite number** n is a nonprime integer; that is, $n = ab$ where a and b are integers with $1 < a < n$ and $1 < b < n$.
- For two numbers x and y, $x < y$ means $y - x > 0$.
- For two integers n and m, n **divides** m, $n \mid m$, means that m is divisible by n—that is, $m = k(n)$ for some integer k.
- The **absolute value** of a number x, $|x|$, is x if $x \geq 0$ and is $-x$ if $x < 0$.

SECTION 2.1 REVIEW

TECHNIQUES

- Look for a counterexample.
- Construct direct proofs, proofs by contraposition, and proofs by contradiction.

MAIN IDEAS

- Inductive reasoning is used to formulate a conjecture based on experience. Deductive reasoning is used either to refute a conjecture by finding a counterexample or to prove a conjecture.
- In proving a conjecture about some subject, facts about that subject can be used.
- Under the right circumstances, proof by contraposition or contradiction may work better than a direct proof.

EXERCISES 2.1

1. Write the contrapositive of each statement in Exercise 5 of Section 1.1.
2. Write the converse of each statement in Exercise 5 of Section 1.1.

3. Provide counterexamples to the following statements.
   a. Every geometric figure with four right angles is a square.
   b. If a real number is not positive, then it must be negative.
   c. All people with red hair have green eyes or are tall.
   d. All people with red hair have green eyes and are tall.
4. Provide counterexamples to the following statements.
   a. If a and b are integers where $a \mid b$ and $b \mid a$, then $a = b$.
   b. If $n^2 > 0$ then $n > 0$.
   c. If n is an even number, then $n^2 + 1$ is prime.
   d. If n is a positive integer, then $n^3 > n!$.
5. Provide a counterexample to the following statement: The number n is an odd integer if and only if $3n + 5$ is an even integer.
6. Provide a counterexample to the following statement: The number n is an even integer if and only if $3n + 2$ is an even integer.
7. a. Find two even integers whose sum is not a multiple of 4.
   b. What is wrong with the following "proof" that the sum of two even numbers is a multiple of 4?
   Let x and y be even numbers. Then $x = 2m$ and $y = 2m$, where m is an integer, so $x + y = 2m + 2m = 4m$, which is an integral multiple of 4.
8. a. Find an example of an odd number x and an even number y such that $x - y = 7$.
   b. What is wrong with the following "proof" that an odd number minus an even number is always 1?
   Let x be odd and y be even. Then $x = 2m + 1$, $y = 2m$, where m is an integer, and $x - y = 2m + 1 - 2m = 1$.

For Exercises 9–46, prove the given statement.

9. If $n = 25$, 100, or 169, then n is a perfect square and is a sum of two perfect squares.
10. If n is an even integer, $4 \leq n \leq 12$, then n is a sum of two prime numbers.
11. For any positive integer n less than or equal to 3, $n! < 2^n$.
12. For $2 \leq n \leq 4$, $n^2 \geq 2^n$.
13. The sum of two even integers is even (do a direct proof).
14. The sum of two even integers is even (do a proof by contradiction).
15. The sum of two odd integers is even.
16. The sum of an even integer and an odd integer is odd.
17. An odd integer minus an even integer is odd.
18. If n is an even integer, then $n^2 - 1$ is odd.
19. The product of any two consecutive integers is even.
20. The sum of an integer and its square is even.
21. The square of an even number is divisible by 4.
22. For every integer n, the number $3(n^2 + 2n + 3) - 2n^2$ is a perfect square.
23. If a number x is positive, so is $x + 1$ (do a proof by contraposition).
24. If n is an odd integer, then it is the difference of two perfect squares.
25. The number n is an odd integer if and only if $3n + 5 = 6k + 8$ for some integer k.
26. The number n is an even integer if and only if $3n + 2 = 6k + 2$ for some integer k.

27. For x and y positive numbers, $x < y$ if and only if $x^2 < y^2$.
28. If $x^2 + 2x - 3 = 0$, then $x \neq 2$.
29. If n is an even prime number, then $n = 2$.
30. The sum of three consecutive integers is divisible by 3.
31. If two integers are each divisible by some integer n, then their sum is divisible by n.
32. If the product of two integers is not divisible by an integer n, then neither integer is divisible by n.
33. If n, m, and p are integers and $n \mid m$ and $m \mid p$, then $n \mid p$.
34. If n, m, p, and q are integers and $n \mid p$ and $m \mid q$, then $nm \mid pq$.
35. The square of an odd integer equals $8k + 1$ for some integer k.
36. The sum of the squares of two odd integers cannot be a perfect square. (*Hint:* Use Exercise 35.)
37. The product of the squares of two integers is a perfect square.
38. The difference of two consecutive cubes is odd.
39. For any two numbers x and y, $|x + y| \leq |x| + |y|$.
40. For any two numbers x and y, $|xy| = |x||y|$.
41. The value A is the average of the n numbers $x_1, x_2, \ldots, x_n$. Prove that at least one of $x_1, x_2, \ldots, x_n$ is greater than or equal to A.
42. Suppose you were to use the steps of Example 11 to attempt to prove that $\sqrt{4}$ is not a rational number. At what point would the proof not be valid?
43. Prove that $\sqrt{3}$ is not a rational number.
44. Prove that $\sqrt{5}$ is not a rational number.
45. Prove that $\sqrt[3]{2}$ is not a rational number.
46. Prove that $\log_2 5$ is not a rational number ($\log_2 5 = x$ means $2^x = 5$).

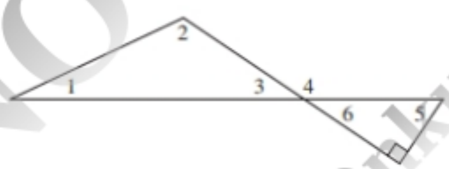
For Exercises 47–72, prove or disprove the given statement.

47. 0 is an even number.
48. 91 is a composite number.
49. 297 is a composite number.
50. 83 is a composite number.
51. The difference between two odd integers is odd.
52. The difference between two even integers is even.
53. The product of any three consecutive integers is even.
54. The sum of any three consecutive integers is even.
55. The sum of an integer and its cube is even.
56. The number n is an even integer if and only if $n^3 + 13$ is odd.
57. The product of an integer and its square is even.
58. Any positive integer can be written as the sum of the squares of two integers.
59. The sum of the square of an odd integer and the square of an even integer is odd.
60. If n is a positive integer that is a perfect square, then $n + 2$ is not a perfect square.
61. For a positive integer n, $n + \frac{1}{n} \geq 2$.

62. If n, m, and p are integers and $n \mid mp$, then $n \mid m$ or $n \mid p$.
63. For every prime number n, $n + 4$ is prime.
64. For every positive integer n, $n^2 + n + 3$ is not prime.
65. For n a positive integer, $n > 2$, $n^2 - 1$ is not prime.
66. For every positive integer n, $2^n + 1$ is prime.
67. For every positive integer n, $n^2 + n + 1$ is prime.
68. For n an even integer, $n > 2$, $2^n - 1$ is not prime.
69. The sum of two rational numbers is rational.
70. The product of two rational numbers is rational.
71. The product of two irrational numbers is irrational.
72. The sum of a rational number and an irrational number is irrational.

For Exercises 73–75, use the following facts from geometry and the accompanying figure.

- The interior angles of a triangle sum to 180°.
- Vertical angles (opposite angles formed when two lines intersect) are the same size.
- A straight angle is 180°.
- A right angle is 90°.



73. Prove that the measure of angle 5 plus the measure of angle 3 is 90°.
74. Prove that the measure of angle 4 is the sum of the measures of angles 1 and 2.
75. If angle 1 and angle 5 are the same size, then angle 2 is a right angle.
76. Prove that the sum of the integers from 1 through 100 is 5050. (*Hint:* Instead of actually adding all the numbers, try to make the same clever observation that the German mathematician Karl Friederick Gauss [1777–1855] made as a schoolchild: Group the numbers into pairs, using 1 and 100, 2 and 99, etc.)

SECTION 2.2 INDUCTION

First Principle of Induction

There is one final proof technique especially useful in computer science. To illustrate how the technique works, imagine that you are climbing an infinitely high ladder. How do you know whether you will be able to reach an arbitrarily high rung? Suppose we make the following two assertions about your climbing abilities:

1. You can reach the first rung.
2. Once you get to a rung, you can always climb to the next one up. (Notice that this assertion is an implication.)