EXAMPLE 26 Solve the recurrence relation

$$T(1) = 3$$
$$T(n) = 2T\left(\frac{n}{2}\right) + 2n$$

This is a match for Equation (16), where $c = 2$ and $g(n) = 2n$. Therefore $g(2^i) = 2(2^i)$. Substituting into Equation (21)—the solution of Equation (16)—gives the following result, where we use the fact that $2^{\log n} = n$.

$$\begin{aligned} T(n) &= 2^{\log n}T(1) + \sum_{i=1}^{\log n} 2^{\log n - i}2(2^i) \\ &= 2^{\log n}(3) + \sum_{i=1}^{\log n} 2^{\log n + 1} \\ &= n(3) + (2^{\log n + 1})\log n \\ &= 3n + (2^{\log n} \cdot 2)\log n \\ &= 3n + 2n \log n \end{aligned}$$

Practice 15 Show that the solution to the recurrence relation

$$S(1) = 1$$
$$S(n) = 2S\left(\frac{n}{2}\right) + 1 \text{ for } n \geq 2, n = 2^m$$

is $2n - 1$. (*Hint:* See Example 15 in Section 2.2 and note that $2^{\log n} = n$.)

SECTION 3.2 REVIEW

TECHNIQUES

- Solve recurrence relations by the expand, guess, and verify technique.
- Solve linear, first-order recurrence relations with constant coefficients by using a solution formula.
- Solve linear, second-order homogeneous recurrence relations with constant coefficients by using the characteristic equation.
- Solve divide-and-conquer recurrence relations by using a solution formula.

MAIN IDEA

- Certain recurrence relations have closed-form solutions.

EXERCISES 3.2

In Exercises 1–12, solve the recurrence relation subject to the basis step.

1. $S(1) = 5$
   $S(n) = S(n - 1) + 5$ for $n \geq 2$
2. $B(1) = 5$
   $B(n) = 3B(n - 1)$ for $n \geq 2$

3. $F(1) = 2$
   $F(n) = 2F(n-1) + 2^n$ for n ≥ 2
4. $T(1) = 1$
   $T(n) = 2T(n-1) + 1$ for $n \geq 2$
   (*Hint:* See Example 15 in Section 2.2.)
5. $A(1) = 1$
   $A(n) = A(n-1) + n$ for $n \geq 2$
   (*Hint:* See Practice 7 in Section 2.2.)
6. $S(1) = 1$
   $S(n) = S(n-1) + (2n-1)$ for $n \geq 2$
   (*Hint:* See Example 14 in Section 2.2.)
7. $T(1) = 1$
   $T(n) = T(n-1) + n^2$ for $n \geq 2$
   (*Hint*: See Exercise 7 in Section 2.2.)
8. $P(1) = 2$
   $P(n) = 2P(n-1) + n2^n$ for $n \geq 2$
   (*Hint:* See Practice 7 in Section 2.2.)
9. $F(1) = 1$
   $F(n) = nF(n-1)$ for $n \geq 2$
10. $S(1) = 1$
   $S(n) = nS(n-1) + n!$ for $n \geq 2$
11. $A(1) = 1$
   $A(n) = 2(n-1)A(n-1)$ for $n \geq 2$
   (*Hint*: 0! is defined to equal 1.)
12. $P(1) = 2$
   $P(n) = 3(n+1)P(n-1)$ for $n \geq 2$
13. At the beginning of this chapter the contractor claimed:

    The material to be stored at the chemical disposal site degrades to inert matter at the rate of 5% per year. Therefore only about one-third of the original active material will remain at the end of 20 years.

    a. Write a recurrence relation $T(n)$ for the amount of active material at the beginning of year n. Assume that $T(1) = X$, a specific but unknown amount.
    b. Solve the recurrence relation.
    c. Compute $T(21)$ to check the contractor's claim; note that the end of 20 years is the beginning of the 21st year.
14. A colony of bats is counted every 2 months. The first four counts are 1200, 1800, 2700, and 4050.
    a. Assuming that this growth rate continues, write a recurrence relation for the number of bats at count n.
    b. Solve the recurrence relation.
    c. What will the 12th count be?
15. Spam e-mail containing a virus is sent to 1,000 e-mail addresses. After 1 second, a recipient machine broadcasts 10 new spam e-mails containing the virus, after which the virus disables itself on that machine.

a. Write a recurrence relation for the number of e-mails sent at the start of the nth second.

b. Solve the recurrence relation.

c. How many e-mails are sent at the end of 20 seconds (that is, at the beginning of the 21st second)?

16. Total natural gas consumption in the state of New Jersey was 614,908 million cubic feet in 2008 and 653,459 million cubic feet in 2010.

a. Assuming a constant annual percentage growth rate r, write a recurrence relation (in terms of r) for the total natural gas consumption in New Jersey in year n.

b. Solve the recurrence relation (in terms of r).

c. Using the given data, compute the value of r.

d. What will be the total natural gas consumption in New Jersey in the year 2020?

17. A loan of $5,000 is charged a 12% annual interest rate. An $80 payment is made each month.

a. Write a recurrence relation for the loan balance remaining at the beginning of month n.

b. Solve the recurrence relation. (See Exercise 27 of Section 2.2 for the formula for the sum of a geometric sequence.)

c. How much is left of the loan balance at the beginning of the 19th month?

18. In an account that pays 3% annually, $1,000 is deposited. At the end of each year, an additional $100 is deposited into the account.

a. Write a recurrence relation for the amount in the account at the beginning of year n.

b. Solve the recurrence relation. (See Exercise 27 of Section 2.2 for the formula for the sum of a geometric sequence.)

c. What is the account worth at the beginning of the 8th year?

19. The shellfish population in a bay is estimated to have a count of about 1,000,000. Studies show that pollution reduces this population by about 2% per year, while other hazards are judged to reduce the population by about 10,000 per year.

a. Write a recurrence relation for the shellfish population at the beginning of year n.

b. Solve the recurrence relation. (See Exercise 27 of Section 2.2 for the formula for the sum of a geometric sequence.)

c. What is the approximate shellfish population at the beginning of year 10?

20. A certain protected species normally doubles its population each month. The initial population is 20, but by the beginning of the next month, 1 specimen has died of an infection. In successive months, the infection kills 2, then 4, then 8, and so forth.

a. Write a recurrence relation for the size of the population at the beginning of month n.

b. Solve this recurrence relation.

c. What is the size of the population at the beginning of month 7?

21. A computer virus that spreads by way of e-mail messages is planted in 3 machines the first day. Each day, each infected machine from the day before infects 5 new machines. By the end of the second day, a software solution has been found to counteract the virus, and 1 machine is clean at that point. Each day thereafter, 6 times as many machines are clean as were clean the day before.

a. Write a recurrence relation for the total number of infected machines on day n.

b. Solve this recurrence relation.

c. How many days will it be before the effects of the virus are completely gone?

22. This problem concerns the Towers of Hanoi puzzle (see Exercise 82 in Section 3.1).

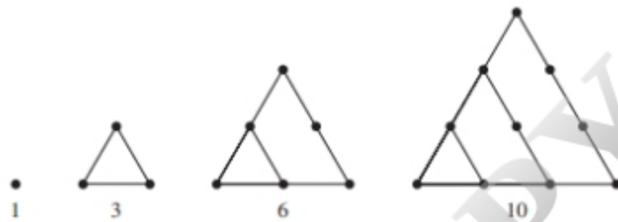
a. Based on the recursive algorithm of Exercise 82 in Section 3.1, find a recurrence relation $M(n)$ for the number of disk moves required to solve the Towers of Hanoi puzzle for n disks.

b. Solve this recurrence relation. (*Hint:* See Exercise 15 in Section 2.2.)

c. Go through the steps of the solution algorithm for $n = 3$ and record the number of disk moves required. Compare this number with the result from part (*b*) with $n = 3$.

d. The mythical origin of the Towers of Hanoi puzzle concerns 64 golden disks that a group of monks are moving from one tower to another. When their task is complete, the world will end. Assuming that the monks can move 1 disk per second, calculate the number of years to complete the task.

23. Early members of the Pythagorean Society defined *figurate numbers* to be the number of dots in certain geometrical configurations. The first few *triangular numbers* are 1, 3, 6, and 10:



Find and solve a recurrence relation for the *n*th triangular number. (*Hint:* See Practice 7 in Section 2.2.)

24. The first few *square numbers* (see the previous Exercise) are 1, 4, 9, and 16:

Find and solve a recurrence relation for the *n*th square number. (*Hint*: See Example 14 in Section 2.2.)

25. The first few *pentagonal numbers* (see Exercise 23) are 1, 5, 12, and 22:

Find and solve a recurrence relation for the *n*th pentagonal number. (*Hint:* See Exercise 28 of Section 2.2 for the formula for the sum of an arithmetic sequence.)

26. Use induction to verify that equation (8) of this section is the solution to the recurrence relation (6) subject to the basis condition that $S(1)$ is known.

In Exercises 27–34, solve the recurrence relation subject to the initial conditions.

27. $T(1) = 5$
$T(2) = 11$
$T(n) = 5T(n-1) - 6T(n-2)$ for $n \geq 3$

28. $A(1) = 7$
 $A(2) = 18$
 $A(n) = 6A(n-1) - 8A(n-2)$ for $n \geq 3$
29. $S(1) = 4$
 $S(2) = -2$
 $S(n) = -S(n-1) + 2S(n-2)$ for $n \geq 3$
30. $P(1) = 5$
 $P(2) = 17$
 $P(n) = 7P(n-1) - 12P(n-2)$ for $n \geq 3$
31. $F(1) = 8$
 $F(2) = 16$
 $F(n) = 6F(n-1) - 5F(n-2)$ for $n \geq 3$
32. $T(1) = -1$
 $T(2) = 7$
 $T(n) = -4T(n-1) - 3T(n-2)$ for $n \geq 3$
33. $B(1) = 3$
 $B(2) = 14$
 $B(n) = 4B(n-1) - 4B(n-2)$ for $n \geq 3$
34. $F(1) = -10$
 $F(2) = 40$
 $F(n) = -10F(n-1) - 25F(n-2)$ for $n \geq 3$

In Exercises 35 and 36, solve the recurrence relation subject to the initial conditions; the solutions involve complex numbers.

35. $A(1) = 8$
 $A(2) = 8$
 $A(n) = 2A(n-1) - 2A(n-2)$ for $n \geq 3$
36. $S(1) = 4$
 $S(2) = -8$
 $S(n) = -4S(n-1) - 5S(n-2)$ for $n \geq 3$
37. Solve the Fibonacci recurrence relation

$$F(1) = 1$$
$$F(2) = 1$$
$$F(n) = F(n-1) + F(n-2) \text{ for } n > 2$$

 Compare your answer with Exercise 31 of Section 3.1.

38. Find a closed-form solution for the Lucas sequence

$$L(1) = 1$$
$$L(2) = 3$$
$$L(n) = L(n-1) + L(n-2) \text{ for } n \geq 3$$

39. Houses in a new development go on sale initially for an average price of \$200,000. At the beginning of month 2, the average sale price has risen to \$250,000. At the beginning of each succeeding month, the average price increase is half what it was the previous month.
    a. Write and solve a recurrence relation for $M(n)$, the average sale price at the beginning of month n.
    b. At the beginning of which month is the average price within \$2,000 of \$300,000?
40. A contaminated soil site is tested monthly for the presence of a particular microorganism. Initially, 950 microorganisms per cubic foot of soil are found; at the beginning of month 2, there are 1,000 organisms per cubic foot. Left untreated, the growth rate of this microorganism increases by 25% per month.
    a. Write and solve a recurrence relation for $O(n)$, the number of organisms present per cubic foot at the beginning of month n.
    b. At the end of what month does the number of organisms first exceed 5,000 per cubic foot?
41. Prove that the number of binary strings of length n with no two consecutive 0*s* is given by the Fibonacci sequence term $F(n + 2)$. (*Hint:* Write a recurrence relation; consider strings of length n that end in 1 and those that end in 0.)
42. a. Find a recurrence relation for the number of binary strings of length n that have two consecutive 1*s*.
    b. How many binary strings of length 4 have two consecutive 1*s*? What are these strings?
43. Consider the recurrence relation $S(n) = c_1S(n - 1)$ as a linear second-order homogeneous recurrence relation with constant coefficients where $c_2 = 0$. Solve this recurrence relation using its characteristic equation, and prove that the solution is the same as that of Equation (8).
44. Prove that

$$S(n) = pr^{n-1} + q(n - 1)r^{n-1}$$

where

$$p = S(1)$$
$$pr + qr = S(2)$$

is a solution to the recurrence relation $S(n) = c_1S(n - 1) + c_2S(n - 2)$ for all $n \geq 1$ if r is a repeated root of the characteristic equation.

In Exercises 45–48, solve the recurrence relation subject to the basis step. (*Hint:* See Example 15 in Section 2.2, and note that $2^{\log n} = n$.)

45. $P(1) = 1$

    $P(n) = 2P\left(\frac{n}{2}\right) + 3$ for $n \geq 2, n = 2^m$
46. $T(1) = 3$

    $T(n) = T\left(\frac{n}{2}\right) + n$ for $n \geq 2, n = 2^m$
47. $S(1) = 1$

    $S(n) = 2S\left(\frac{n}{2}\right) + n$ for $n \geq 2, n = 2^m$
48. $P(1) = 1$

    $P(n) = 2P\left(\frac{n}{2}\right) + n^2$ for $n \geq 2, n = 2^m$