SECTION 6.1 REVIEW

TECHNIQUES

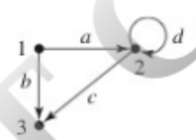
- Use graph terminology.
- Prove or disprove that two graphs are isomorphic.
- Find a planar representation of a simple graph or prove that none exists.
- Construct adjacency matrices and adjacency lists for graphs and directed graphs.

MAIN IDEAS

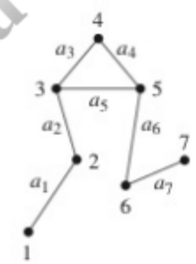
- Diverse situations can be modeled by graphs.
- Graphs can be represented in a computer by matrices or by linked lists.

EXERCISES 6.1

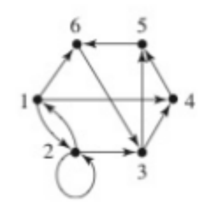
1. Give the function g that is part of the formal definition of the directed graph shown.



2. Use the graph in the figure to answer the questions that follow.
   a. Is the graph simple?
   b. Is the graph complete?
   c. Is the graph connected?
   d. Can you find two paths from 3 to 6?
   e. Can you find a cycle?
   f. Can you find an arc whose removal will make the graph acyclic?
   g. Can you find an arc whose removal will make the graph not connected?

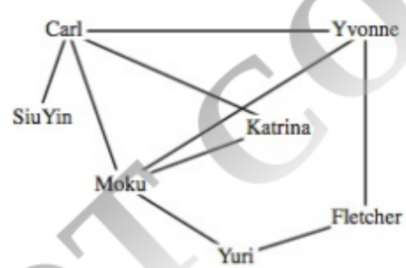


3. Sketch a picture of each of the following graphs.
   a. Simple graph with three nodes, each of degree 2
   b. Graph with four nodes, with cycles of length 1, 2, 3, and 4
   c. Noncomplete graph with four nodes, each of degree 4
4. Use the directed graph in the figure to answer the questions that follow.
   a. Which nodes are reachable from node 3?
   b. What is the length of the shortest path from node 3 to node 6?
   c. What is a path from node 1 to node 6 of length 8?



5. Draw K_6.
6. Draw $K_{3,4}$.
7. For each of the following characteristics, draw a graph or explain why such a graph does not exist.
   a. Four nodes of degree 1, 2, 3, and 4, respectively
   b. Simple, four nodes of degree 1, 2, 3, and 4, respectively
   c. Four nodes of degree 2, 3, 3, and 4, respectively
   d. Four nodes of degree 2, 3, 3, and 3, respectively

8. For each of the following characteristics, draw a graph or explain why such a graph does not exist.
   a. Simple graph with seven nodes, each of degree 3
   b. Four nodes, two of degree 2 and two of degree 3
   c. Three nodes of degree 0, 1, and 3, respectively
   d. Complete graph with 4 nodes each of degree 2
9. An *acquaintanceship graph* is an undirected graph in which the nodes represent people and nodes *a* and *b* are adjacent if *a* and *b* are acquainted.
   a. The acquaintanceship graph for the IT department and the marketing department of a major corporation is an unconnected graph. What does this imply?
   b. The following figure represents an acquaintanceship graph for residents of an apartment building. Are Carl and Fletcher acquainted? How many people is SiuYin acquainted with?



   c. The length of the shortest path between node *a* and node *b* in an acquaintanceship graph is sometimes called the *degree of separation* between *a* and *b*. What is the degree of separation between Carl and Yuri?
10. The "small world effect" states that the average degree of separation (see Exercise 9) in an acquaintanceship graph of the whole world is 6. In other words, a path of acquaintance relationships from you to any other person on earth exists with, on the average, a path length of 6 (5 intermediate persons). Experiments in delivering hard-copy letters and e-mail messages have empirically confirmed this theory.[1]
    a. What are the potential implications for e-mail traffic if the small world effect holds for computer networks?
    b. What are the potential implications for epidemiology if the small world effect holds for physical contact between humans?
11. The small world effect (see Exercise 10) has been found to be true between root words (that is, basic words found in a thesaurus) in the English language, with an average degree of separation equal to 3. Here "adjacent words" are those that are listed as synonyms in an English thesaurus. For example, "gate" and "commotion" are related by 3 degrees of separation, as follows:

    gate → door → flap → commotion

    Can you think of 3 degrees of separation between the following pairs of words?
    a. "star" and "sculpture"
    b. "burden" and "influence"
    c. "piano" and "significance"

[1]But more recent analyses of 721 million Facebook users, a much larger community than was available to earlier studies, suggests that the average number of intermediaries between persons *A* and *B* is 3.74. It's a small world indeed, at least for Facebook users.

12. An idea closely related to the average degree of separation in a graph is that of clustering. The *global clustering coefficient* for a given graph is given by

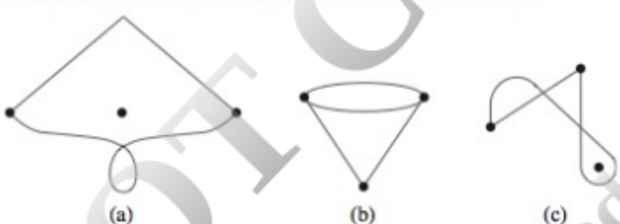
$$C = \frac{3 * T}{t}$$

where T = the number of triangles in the graph and t = the number of connected node triples.

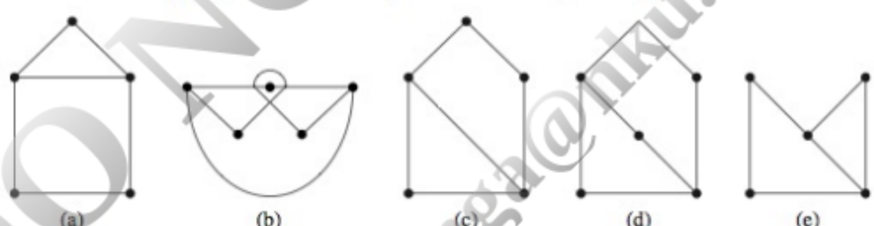
A connected node triple is a "center" node adjacent to an unordered pair of other nodes. For example, in the graph of Exercise 2, 3–4–5 (or 5–4–3) and 4–5–6 (6–5–4) are two such triples. Nodes that make up a triangle demonstrate transitivity; if a is adjacent to b and b is adjacent to c, then a is adjacent to c. Therefore c is a ratio of nodes in a transitive threesome to all nodes in a threesome. (One might think of this in terms of a social network as the probability that if you are a "friend" of mine and x is a "friend" of yours, then x is also a "friend" of mine.)

a. Consider the graph in Figure 6.28 and the graph for Exercise 2. Which do you think has the higher clustering coefficient?
b. Compute the clustering coefficient for the graph in Figure 6.28
c. Compute the clustering coefficient for the graph for Exercise 2.

13. Which of the following graphs is not isomorphic to the others, and why?

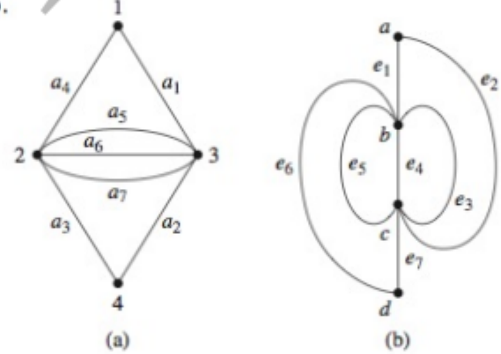


14. Which of the following graphs is not isomorphic to the others, and why?

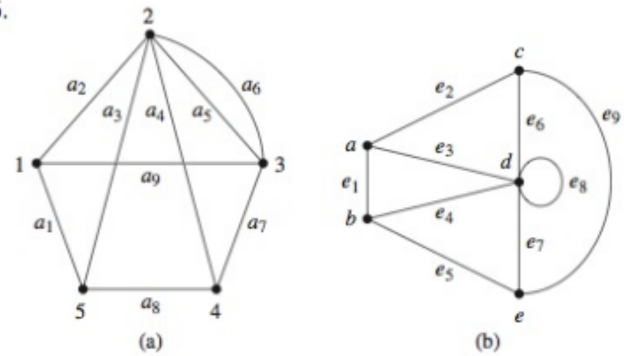


For Exercises 15–20, decide if the two graphs are isomorphic. If so, give the function or functions that establish the isomorphism; if not, explain why.

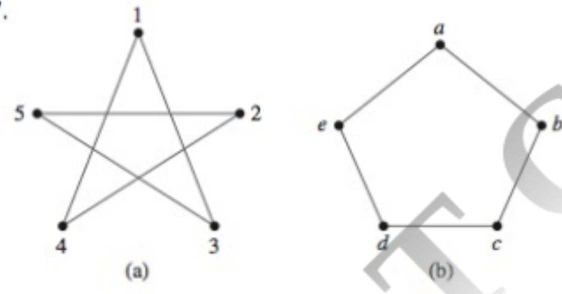
15.



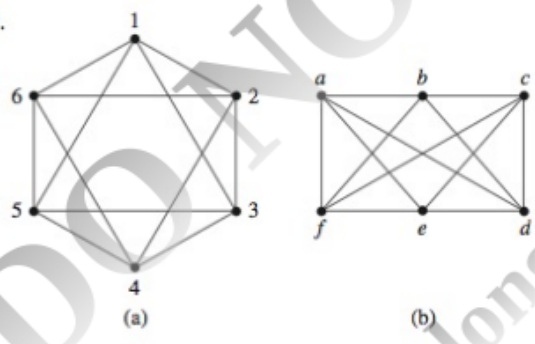
16.


2
a_2 a_3 a_4 a_5 a_6
1 a_9 3
a_1 a_7
5 a_8 4
(a)
c
e_2 e_6 e_9
a e_3 d
e_1 e_8
b e_4
e_7
e_5
e
(b)

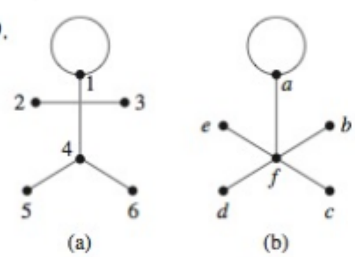
17.


1
5 2
4 3
(a)
a
e b
d c
(b)

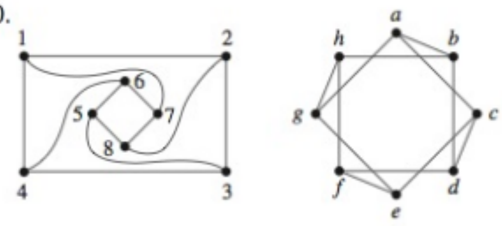
18.


1
6 2
5 3
4
(a)
a b c
f e d
(b)

19.


1
2 3
4
5 6
(a)
a
e b
f
d c
(b)

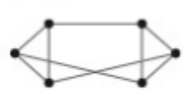
20.



(a) (b)

21. Prove that two graphs are not isomorphic if one of them
   a. has more nodes than the other.
   b. has more arcs than the other.
   c. has parallel arcs and the other does not.
   d. has a loop and the other does not.
   e. has a node of degree k and the other does not.
   f. is connected and the other is not.
   g. has a cycle and the other does not.
22. Draw all the nonisomorphic, simple graphs with two nodes.
23. Draw all the nonisomorphic, simple graphs with three nodes.
24. Draw all the nonisomorphic, simple graphs with four nodes.
25. Find an expression for the number of arcs in K_n and prove that your expression is correct.
26. Verify Euler's formula for the following simple, connected, planar graph.

27. Prove that $K_{2,3}$ is a planar graph.
28. Prove that the following graph is a planar graph.

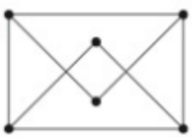


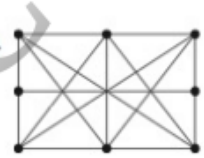
29. If a simple, connected, planar graph has six nodes, all of degree 3, into how many regions does it divide the plane?

30. If all the nodes of a simple, connected, planar graph have degree 4 and the number of arcs is 12, into how many regions does it divide the plane?
31. Does Euler's formula (Equation (1) of the theorem on the number of nodes and arcs) hold for nonsimple graphs? What about inequalities (2) and (3) of the theorem?
32. What is wrong with the following argument that claims to use elementary subdivisions to turn a nonplanar graph into a planar graph?

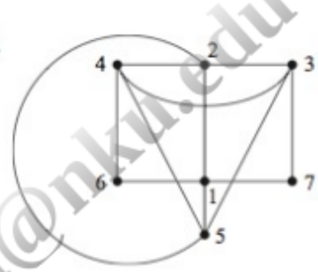
    In a nonplanar graph there must be two arcs a_i and a_j that intersect at a point v that is not a node. Do an elementary subdivision on a_i with an inserted node at v and an elementary subdivision on a_j with an inserted node at v. In the resulting graph, the point of intersection is a node. Repeat this process with any non-node intersections; the result is a planar graph.

For Exercises 33–36, determine whether the graph is planar (by finding a representation where arcs intersect only at nodes) or nonplanar (by finding a subgraph homeomorphic to K_5 or $K_{3,3}$.

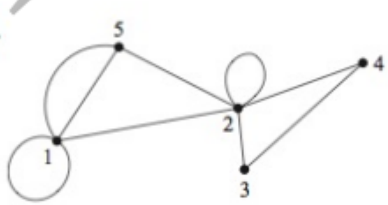
33.


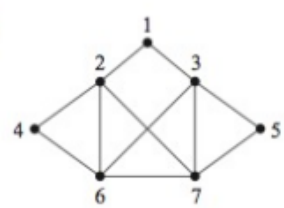
35.


34.


36.


For Exercises 37–42, write the adjacency matrix for the given graph.

37.


39.


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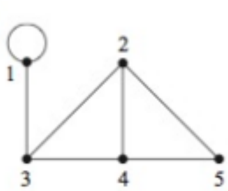

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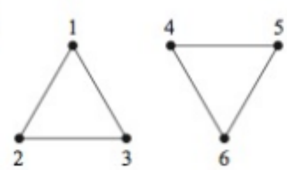
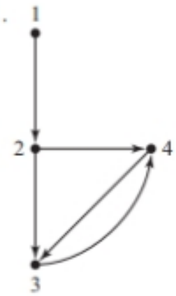
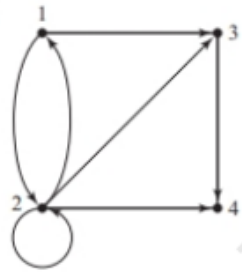


41.



42.



For Exercises 43–46, draw the graph represented by the adjacency matrix.

43. $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

44. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

45. $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

46. $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

47. The adjacency matrix for an undirected graph is given in lower triangular form by

$$\begin{bmatrix} 2 & & & \\ 1 & 0 & & \\ 0 & 1 & 1 & \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Draw the graph.

48. The adjacency matrix for a directed graph is given by

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Draw the graph.

49. Describe the graph whose adjacency matrix is I_n, the $n \times n$ identity matrix.

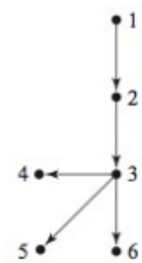
50. Describe the graph whose adjacency matrix is 0_n, the $n \times n$ matrix of all 0's.

51. Describe the adjacency matrix for K_n, the simple, complete graph with n nodes.

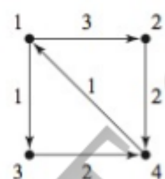
52. Given the adjacency matrix **A** for a directed graph G, describe the graph represented by the adjacency matrix $\mathbf{A}^T$ (see Exercise 15 in Section 5.7).

For Exercises 53–58, draw the adjacency list representation for the indicated graph.

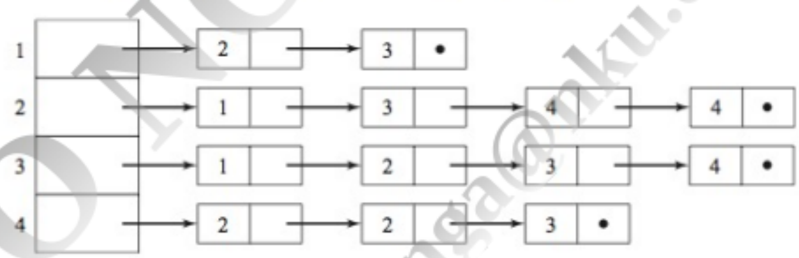
53. Exercise 37
54. Exercise 38
55. Exercise 39
56. Exercise 40
57. Exercise 41
58. Exercise 42
59. Refer to the accompanying graph.
    a. Draw the adjacency list representation.
    b. How many storage locations are required for the adjacency list? (A pointer takes one storage location.)
    c. How many storage locations would be required in an adjacency matrix for this graph?



60. Draw the adjacency list representation for the following weighted directed graph.



61. For the directed graph of Exercise 42, construct the array-pointer representation.
62. For the weighted directed graph of Exercise 60, construct the array-pointer representation.
63. Draw the undirected graph represented by the following adjacency list.



64. Draw the directed graph represented by the following adjacency list.