SECTION 1.3 REVIEW

TECHNIQUES

- Determine the truth value of a predicate wff in a given interpretation.
- Translate English language statements into predicate wffs, and vice versa.
- Recognize a valid wff and explain why it is valid.
- Recognize a nonvalid wff and construct an interpretation in which it is false or has no truth value.

MAIN IDEAS

- The truth value of predicate wffs depends on the interpretation considered.
- Valid predicate wffs are "intrinsically true"—true in all interpretations.

EXERCISES 1.3

1. What is the truth value of each of the following wffs in the interpretation where the domain consists of the integers, $O(x)$ is "x is odd," $L(x)$ is "$x < 10$," and $G(x)$ is "$x > 9$"?
   a. $(\exists x)O(x)$
   b. $(\forall x)[L(x) \rightarrow O(x)]$
   c. $(\exists x)[L(x) \wedge G(x)]$
   d. $(\forall x)[L(x) \vee G(x)]$
2. What is the truth value of each of the following wffs in the interpretation where the domain consists of the integers, $A(x)$ is "$x < 5$" and $B(x)$ is "$x < 7$"?
   a. $(\exists x)A(x)$
   b. $(\exists x)[A(x) \wedge B(x)]$
   c. $(\forall x)[A(x) \rightarrow B(x)]$
   d. $(\forall x)[B(x) \rightarrow A(x)]$
3. What is the truth value of each of the following wffs in the interpretation where the domain consists of the integers?
   a. $(\forall x)(\exists y)(x + y = x)$
   b. $(\exists y)(\forall x)(x + y = x)$
   c. $(\forall x)(\exists y)(x + y = 0)$
   d. $(\exists y)(\forall x)(x + y = 0)$
   e. $(\forall x)(\forall y)(x < y \vee y < x)$
   f. $(\forall x)[x < 0 \rightarrow (\exists y)(y > 0 \wedge x + y = 0)]$
   g. $(\exists x)(\exists y)(x^2 = y)$
   h. $(\forall x)(x^2 > 0)$
4. What is the truth value of each of the following wffs in the interpretation where the domain consists of the real numbers?
   a. $(\forall x)(\exists y)(x = y^2)$
   b. $(\forall x)(\forall y)(x = y^2)$
   c. $(\exists x)(\forall y)(x = y^2)$
   d. $(\exists x)(\exists y)(x = y^2)$
5. Give the truth value of each of the following wffs in the interpretation where the domain consists of the states of the United States, $Q(x, y)$ is "x is north of y," $P(x)$ is "x starts with the letter M," and a is "Massachusetts."
   a. $(\forall x)P(x)$
   b. $(\forall x)(\forall y)(\forall z)[Q(x, y) \wedge Q(y, z) \rightarrow Q(x, z)]$
   c. $(\exists y)(\exists x)Q(y, x)$
   d. $(\forall x)(\exists y)[P(y) \wedge Q(x, y)]$
   e. $(\exists y)Q(a, y)$
   f. $(\exists x)[P(x) \wedge Q(x, a)]$
6. Give the truth value of each of the following wffs in the interpretation where the domain consists of people, $M(x, y)$ is "x is the mother of y", $F(x)$ is "x is female", $M(x)$ is "x is male."
   a. $(\forall x)(\exists y)(M(y, x))$
   b. $(\exists x)(\forall y)(M(x, y))$
   c. $(\forall x)(\forall y)(M(x, y) \rightarrow M(y))$
   d. $(\exists x)(\exists y)(M(x, y) \wedge M(y))$
   e. $(\exists x)(\forall y)(M(x, y) \rightarrow F(y))$

7. For each wff, find an interpretation in which it is true and one in which it is false.
   a. $(\forall x)([A(x) \vee B(x)] \wedge [A(x) \wedge B(x)]')$
   b. $(\forall x)(\forall y)[P(x, y) \rightarrow P(y, x)]$
   c. $(\forall x)[P(x) \rightarrow (\exists y)Q(x, y)]$
8. For each wff, find an interpretation in which it is true and one in which it is false.
   a. $(\exists x)[A(x) \wedge (\forall y)B(x, y)]$
   b. $[(\forall x)A(x) \rightarrow (\forall x)B(x)] \rightarrow (\forall x)[A(x) \rightarrow B(x)]$
   c. $(\exists x)[P(x) \vee Q(x)] \wedge (\forall x)[P(x) \rightarrow Q(x)]$
9. Identify the scope of each of the quantifiers in the following wffs and indicate any free variables.
   a. $(\forall x)[P(x) \rightarrow Q(y)]$
   b. $(\exists x)[A(x) \wedge (\forall y)B(y)]$
   c. $(\exists x)[(\forall y)P(x, y) \wedge Q(x, y)]$
   d. $(\exists x)(\exists y)[A(x, y) \wedge B(y, z) \rightarrow A(a, z)]$
10. Explain why each of the following expressions is written incorrectly.
    a. $(\exists)(Q(x) \wedge P(x)$
    b. $(\forall y)(Q(y)\ P(y))$
    c. $(\forall x)(\forall y)Q(x) \rightarrow P(y)$
11. Which of the following sentences are equivalent to the statement

    All circles are round.

    a. If it's round, it's a circle.
    b. Roundness is a necessary property of circles.
    c. Something that isn't round can't be a circle.
    d. Some round things are circles.
12. Which of the following sentences are equivalent to the statement

    Cats are smarter than dogs.

    a. Some cats are smarter than some dogs.
    b. There is a cat that is smarter than all dogs.
    c. All cats are smarter than all dogs.
    d. Only cats are smarter than dogs.
    e. All cats are smarter than any dog.
13. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

    $D(x)$: x is a day
    $S(x)$: x is sunny
    $R(x)$: x is rainy
    M: Monday
    T: Tuesday
    a. All days are sunny.
    b. Some days are not rainy.
    c. Every day that is sunny is not rainy.
    d. Some days are sunny and rainy.

e. No day is both sunny and rainy.
f. It is always a sunny day only if it is a rainy day.
g. No day is sunny.
h. Monday was sunny; therefore, every day will be sunny.
i. It rained both Monday and Tuesday.
j. If some day is rainy, then every day will be sunny.

14. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$B(x)$: x is a ball
$R(x)$: x is round
$S(x)$: x is a soccer ball

a. All balls are round.
b. Not all balls are soccer balls.
c. All soccer balls are round.
d. Some balls are not round.
e. Some balls are round but soccer balls are not.
f. Every round ball is a soccer ball.
g. Only soccer balls are round balls.
h. If soccer balls are round, then all balls are round.

15. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$M(x)$: x is a man
$W(x)$: x is a woman
$T(x)$: x is tall

a. All men are tall.
b. Some women are tall.
c. All men are tall but no woman is tall.
d. Only women are tall
e. No man is tall.
f. If every man is tall, then every woman is tall.
g. Some woman is not tall.
h. If no man is tall, then some woman is not tall.

16. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$A(x)$: x is an animal
$B(x)$: x is a bear
$H(x)$: x is hungry
$W(x)$: x is a wolf

a. Bears are animals.
b. No wolf is a bear.

c. Only bears are hungry.
d. If all wolves are hungry, so are bears.
e. Some animals are hungry bears.
f. Bears are hungry but some wolves are not.
g. If wolves and bears are hungry, so are all animals.
h. Some wolves are hungry but not every animal is hungry.

17. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$P(x)$: x is a person
$T(x)$: x is a time
$F(x, y)$: x is fooled at y

a. You can fool some of the people all of the time.
b. You can fool all of the people some of the time.
c. You can't fool all of the people all of the time.

18. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$L(x)$: x is a lion
$R(x)$: x roars
$P(x)$: x is a predator
$Z(x)$: x is a zebra
$E(x, y)$: x eats y

a. All lions are predators.
b. Some lions roar.
c. Only lions roar.
d. Some lions eat all zebras.
e. All lions eat all zebras.

19. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$G(x)$: x is a game
$M(x)$: x is a movie
$F(x, y)$: x is more fun than y

a. Any movie is more fun than any game.
b. No game is more fun than every movie.
c. Only games are more fun than movies.
d. All games are more fun than some movie.

20. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$C(x)$: x is a child
$T(x)$: x is a toy

$V(x)$: x is a vegetable
$W(x, y)$: x wants y

a. Every child wants toys.
b. Only children want toys.
c. Some child wants only toys.
d. No child wants vegetables.

21. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$J(x)$: x is a judge
$L(x)$: x is a lawyer
$W(x)$: x is a woman
$C(x)$: x is a chemist
$A(x, y)$: x admires y

a. There are some women lawyers who are chemists.
b. No woman is both a lawyer and a chemist.
c. Some lawyers admire only judges.
d. All judges admire only judges.
e. Only judges admire judges.
f. All women lawyers admire some judge.
g. Some women admire no lawyer.

22. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$C(x)$: x is a Corvette
$F(x)$: x is a Ferrari
$P(x)$: x is a Porsche
$S(x, y)$: x is slower than y

a. Nothing is both a Corvette and a Ferrari.
b. Some Porsches are slower than only Ferraris.
c. Only Corvettes are slower than Porsches.
d. All Ferraris are slower than some Corvettes.
e. Some Porsches are slower than no Corvette.
f. If there is a Corvette that is slower than a Ferrari, then all Corvettes are slower than all Ferraris.

23. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$B(x)$: x is a bee
$F(x)$: x is a flower
$L(x, y)$: x loves y

a. All bees love all flowers.
b. Some bees love all flowers.
c. All bees love some flowers.
d. Every bee hates only flowers.
e. Only bees love flowers.
f. Every bee loves only flowers.

g. No bee loves only flowers.
h. Some bees love some flowers.
i. Some bees love only flowers.
j. Every bee hates some flowers.
k. Every bee hates all flowers.
l. No bee hates all flowers.

24. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$S(x)$: x is a spy novel
$L(x)$: x is long
$M(x)$: x is a mystery
$B(x, y)$: x is better than y

a. All spy novels are long.
b. Not every mystery is a spy novel.
c. Only mysteries are long.
d. Some spy novels are mysteries.
e. Spy novels are better than mysteries.
f. Some mysteries are better than all spy novels.
g. Only spy novels are better than mysteries.

25. Give English language translations of the following wffs if

$L(x, y)$: x loves y
$H(x)$: x is handsome
$M(x)$: x is a man
$P(x)$: x is pretty
$W(x)$: x is a woman
j: John
k: Kathy
$L(x, y)$: x loves y

a. $H(j) \wedge L(k, j)$
b. $(\forall x)[M(x) \rightarrow H(x)]$
c. $(\forall x)(W(x) \rightarrow (\forall y)[L(x, y) \rightarrow M(y) \wedge H(y)])$
d. $(\exists x)[M(x) \wedge H(x) \wedge L(x, k)]$
e. $(\exists x)(W(x) \wedge P(x) \wedge (\forall y)[L(x, y) \rightarrow H(y) \wedge M(y)])$
f. $(\forall x)[W(x) \wedge P(x) \rightarrow L(j, x)]$

26. Give English language translations of the following wffs if

$M(x)$: x is a man
$W(x)$: x is a woman
i: Ivan
p: Peter
$W(x, y)$: x works for y

a. $(\exists x)(W(x) \wedge (\forall y)(M(y) \rightarrow [W(x, y)]'))$
b. $(\forall x)[M(x) \rightarrow (\exists y)(W(y) \wedge W(x, y))]$
c. $(\forall x)[M(x) \rightarrow (\forall y)(W(x, y) \rightarrow W(y))]$

d. $(\forall x)(\forall y)(M(x) \wedge W(y, x) \rightarrow W(y))$
e. $W(i, p) \wedge (\forall x)[W(p, x) \rightarrow (W(x))'])$
f. $(\forall x)[W(x, i) \rightarrow (W(x))']$

27. Three forms of negation are given for each statement. Which is correct?
    a. Some people like mathematics.
        1. Some people dislike mathematics.
        2. Everybody dislikes mathematics.
        3. Everybody likes mathematics.
    b. Everyone loves ice cream.
        1. No one loves ice cream.
        2. Everyone dislikes ice cream.
        3. Someone doesn't love ice cream.
    c. All people are tall and thin.
        1. Someone is short and fat.
        2. No one is tall and thin.
        3. Someone is short or fat.
    d. Some pictures are old or faded.
        1. Every picture is neither old nor faded.
        2. Some pictures are not old or faded.
        3. All pictures are not old or not faded.
28. Three forms of negation are given for each statement. Which is correct?
    a. Nobody is perfect.
        1. Everyone is imperfect.
        2. Everyone is perfect.
        3. Someone is perfect.
    b. All swimmers are tall.
        1. Some swimmer is not tall.
        2. There are no tall swimmers.
        3. Every swimmer is short.
    c. Every planet is cold and lifeless.
        1. No planet is cold and lifeless.
        2. Some planet is not cold and not lifeless.
        3. Some planet is not cold or not lifeless.
    d. No bears are hungry.
        1. Only bears are hungy.
        2. All bears are hungry.
        3. There is a hungry bear.
29. Write the negation of each of the following statements.
    a. Some Web sites feature audio.
    b. Every Web site has both audio and video.
    c. Every Web site has either audio or video.

d. Some Web sites have neither audio nor video.
e. Every Web site either has text or else has both audio and video.

30. Write the negation of each of the following statements.
a. Only students eat pizza.
b. Every student eats pizza
c. Some students eat only pizza.

31. Write the negation of each of the following statements.
a. Some farmer grows only corn.
b. All farmers grow corn.
c. Corn is grown only by farmers.

32. Write the negation of each of the following statements
a. Some child fears all clowns.
b. Some children fear only clowns.
c. No clown fears any child.

33. Explain why each wff is valid.
a. $(\forall x)(\forall y)A(x, y) \leftrightarrow (\forall y)(\forall x)A(x, y)$
b. $(\exists x)(\exists y)A(x, y) \leftrightarrow (\exists y)(\exists x)A(x, y)$
c. $(\exists x)(\forall y)P(x, y) \rightarrow (\forall y)(\exists x)P(x, y)$
d. $A(a) \rightarrow (\exists x)A(x)$
e. $(\forall x)[A(x) \rightarrow B(x)] \rightarrow [(\forall x)A(x) \rightarrow (\forall x)B(x)]$

34. Give interpretations to prove that each of the following wffs is not valid:
a. $(\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)[A(x) \wedge B(x)]$
b. $(\forall x)(\exists y)P(x, y) \rightarrow (\exists x)(\forall y)P(x, y)$
c. $(\forall x)[P(x) \rightarrow Q(x)] \rightarrow [(\exists x)P(x) \rightarrow (\forall x)Q(x)]$
d. $(\forall x)[A(x)]' \leftrightarrow [(\forall x)A(x)]'$

35. Decide whether each of the following wffs is valid or invalid. Justify your answer.
a. $(\exists x)A(x) \leftrightarrow ((\forall x)[A(x)]')'$
b. $(\forall x)P(x) \vee (\exists x)Q(x) \rightarrow (\forall x)[P(x) \vee Q(x)]$

36. Decide whether each of the following wffs is valid or invalid. Justify your answer.
a. $(\forall x)A(x) \rightarrow ((\exists x)[A(x)]')'$
b. $(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists x)[P(x) \vee Q(x)] \rightarrow (\exists x)[P(x) \wedge Q(x)]$
c. $(\forall x)[P(x) \vee Q(x)] \rightarrow (\forall x)P(x) \vee (\exists y)Q(y)$

37. From Example 24c, we know that $(\forall x)[P(x) \wedge Q(x)] \leftrightarrow (\forall x)P(x) \wedge (\forall x)Q(x)$ is valid. From Practice 21, we know that $(\forall x)[P(x) \vee Q(x)] \leftrightarrow (\forall x)P(x) \vee (\forall x)Q(x)$ is not valid. From Exercise 34a, we know that $(\exists x)[P(x) \wedge Q(x)] \leftrightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$ is not valid. Explain why $(\exists x)[P(x) \vee Q(x)] \leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$ is valid.

38. A predicate wff is in *prenex normal form* if all the quantifiers appear at the front of the wff. Write each of the following expressions as an equivalent wff in prenex normal form.
a. $(\forall x)P(x) \wedge (\forall y)Q(y)$
b. $(\forall x)(P(x) \rightarrow (\forall y)[Q(y) \rightarrow W(x, y)])$
c. $(\exists x)P(x) \wedge (\exists x)Q(x)$