salesman problem. Once again it can be solved using trial and error by tracing all possible paths and keeping track of the weights of those paths that are Hamiltonian circuits, but, again, this is not an efficient algorithm. (Incidentally, the traveling salesman problem for visiting all 48 capitals of the contiguous United States has been solved—a total of 10,628 miles is required!)

SECTION 7.2 REVIEW

TECHNIQUE

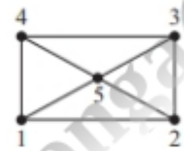
- Using algorithm *EulerPath*, determine whether an Euler path exists in a graph.

MAIN IDEAS

- There is a simple criterion for determining whether Euler paths exist in a graph but no such criterion for whether Hamiltonian circuits exist.
- An algorithm that is $\Theta(n^2)$ in the worst case can determine the existence of an Euler path in a connected graph with n nodes.

EXERCISES 7.2

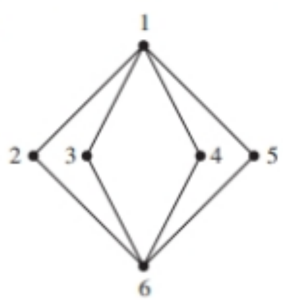
1. Rework Example 3 of Chapter 2 using the theorem on Euler paths. Here is the graph, where the nodes have been numbered.



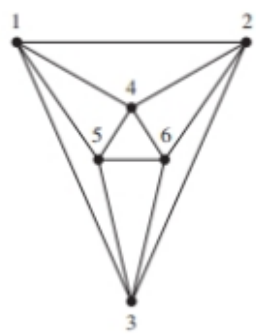
2. a. Add a single arc to the graph of Exercise 1 so that there is an Euler path.
   b. List the nodes in such a path.

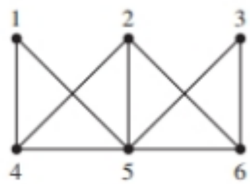
For Exercises 3–12, determine whether the given graph has an Euler path by using the theorem on Euler paths. If so, list the nodes in such a path.

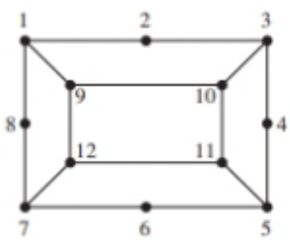
3.

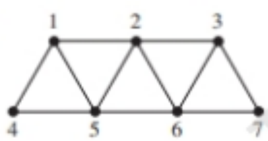


4.

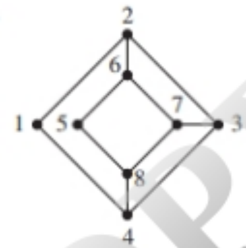


5. 

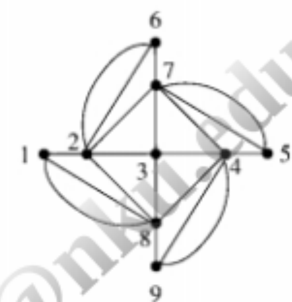
6. 

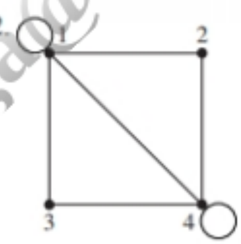
7. 

8. 

9. 

10. 

11. 

12. 

13. Draw the adjacency matrix for the graph of Exercise 3. In applying algorithm *EulerPath*, what is the value of *total* after the second pass through the **while** loop?

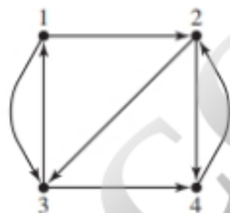
14. Draw the adjacency matrix for the graph of Exercise 5. In applying algorithm *EulerPath*, what is the value of *total* after the fourth pass through the **while** loop?

15. Draw the adjacency matrix for the graph of Exercise 7. In applying algorithm *EulerPath*, what is the value of *i* after the **while** loop is exited?

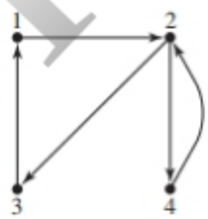
16. Draw the adjacency matrix for the graph of Exercise 9. In applying algorithm *EulerPath*, what is the value of *i* after the **while** loop is exited?

The definition of an Euler path extends to directed graphs. Instead of just the degree of a node as the total number of arc ends, we must now keep track of arcs coming into a node and arcs leaving a node. The total number of arc ends coming into a node is its *in-degree*; the total number of arc ends leaving a node is its *out-degree*. Exercises 17–20 talk about Euler paths in directed graphs.

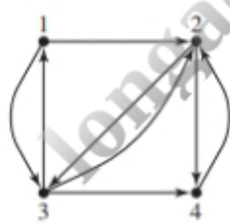
17. Describe two conditions on a connected directed graph, either of which would guarantee the existence of an Euler path.
18. Determine whether this graph has an Euler path. If so, list the nodes in such a path.



19. Determine whether this graph has an Euler path. If so, list the nodes in such a path.



20. Determine whether this graph has an Euler path. If so, list the nodes in such a path.



For Exercises 21–28, decide by trial and error whether Hamiltonian circuits exist for the graphs of the given exercise. If so, list the nodes in such a cycle.

21. Exercise 3
22. Exercise 4
23. Exercise 5
24. Exercise 6
25. Exercise 7
26. Exercise 8
27. Exercise 9
28. Exercise 11
29. Prove that any graph with a Hamiltonian circuit is connected.
30. Find an example of an unconnected graph that has an Euler path. (*Hint*: Because this seems intuitively contradictory, you should look for a trivial case.)

31. Consider a simple, complete graph with n nodes. Testing for a Hamiltonian circuit by trial and error could be done by selecting a fixed starting node and then generating all possible paths from that node of length n.
    a. How many paths of length n are there if repetition of arcs and nodes is allowed?
    b. How many paths of length n are there if repetition of arcs and nodes is allowed but an arc may not be used twice in succession?
    c. How many paths of length n are there if nodes and arcs cannot be repeated except for the starting node? (These are the Hamiltonian circuits.)
    d. To solve the traveling salesman problem in a weighted graph, assume a fixed starting point at node 1 and generate all possible Hamiltonian circuits of length n to find one with minimum weight. If it takes 0.000001 seconds to generate a single Hamiltonian circuit, how long will this process take in a simple, complete graph with 15 nodes?
32. Is it possible to walk in and out of each room in the house shown in the following figure so that each door of the house is used exactly once? Why or why not?

33. Recall that K_n denotes the simple, complete graph of order n.
    a. For what values of n does an Euler path exist in K_n?
    b. For what values of n does a Hamiltonian circuit exist in K_n?
34. Recall that $K_{m,n}$ denotes a bipartite, complete graph with $m + n$ nodes.
    a. For what values of m and n does an Euler path exist in $K_{m,n}$?
    b. For what values of m and n does a Hamiltonian circuit exist in $K_{m,n}$?
35. Prove that a Hamiltonian circuit always exists in a connected graph where every node has degree 2.
36. Consider a connected graph with $2n$ odd vertices, $n \geq 2$. By the theorem on Euler paths, an Euler path does not exist for this graph.
    a. What is the minimum number of disjoint Euler paths, each traveling some of the arcs of the graph, necessary to travel each arc exactly once?
    b. Show that the minimum number is sufficient.
37. Ore's theorem (Oystein Ore, 1960) states that a Hamiltonian circuit exists in any graph G with the following properties:
    1. G is a simple graph with n nodes, $n \geq 3$.
    2. For any two nonadjacent nodes x and y, degree(x) + degree(y) $\geq n$.

    Ore's Theorem is proved by contradiction in the following steps.
    a. Assume that a graph G with properties 1 and 2 above does not have a Hamiltonian circuit. Beginning with G, add new edges to produce a simple graph H that does not have a Hamiltonian circuit but would have such a circuit with the addition of any single new arc. Describe a process for creating H.
    b. Prove that H has a Hamiltonian path, that is, a path that visits each node exactly once.
    c. Denote the nodes on the Hamiltonian path by $p = x_1, x_2, x_3, \ldots, x_{n-1}, x_n = q$. Prove that for any node $x_i, 2 \leq i \leq n-1$, if an arc exists in H between x_i and p, then no arc exists in H between x_{i-1} and q.