really the only kinds of finite Boolean algebras. In a sense we have come full circle. We defined a Boolean algebra to represent many kinds of situations; now we find that (for the finite case) the situations, except for the labels of objects, are the same anyway!

SECTION 8.1 REVIEW

TECHNIQUES

- Decide whether something is a Boolean algebra.
- Prove properties about Boolean algebras.
- Write the equation meaning that a function f preserves an operation from one instance of a structure to another, and verify or disprove such an equation.

MAIN IDEAS

- Mathematical structures serve as models or abstractions of common properties found in diverse situations.
- If there is an isomorphism (a bijection that preserves properties) from A to B, where A and B are instances of a structure, then except for labels, A and B are the same.
- All finite Boolean algebras are isomorphic to Boolean algebras that are power sets.

EXERCISES 8.1

1. Let $B = \{0, 1, a, a'\}$, and let $+$ and $\cdot$ be binary operations on B. The unary operation $'$ is defined by the table

$'$	
0	1
1	0
a	a'
a'	a

Suppose you know that $[B, +, \cdot, ', 0, 1]$ is a Boolean algebra. Making use of the properties that must hold in any Boolean algebra, fill in the following tables defining the binary operations $+$ and $\cdot$:

$+$	0	1	a	a'
0				
1				
a				
a'				

$\cdot$	0	1	a	a'
0				
1				
a				
a'				

2. a. What does the universal bound property (Practice 3) become in the context of propositional logic?
   b. What does it become in the context of set theory?
3. Define two binary operations $+$ and $\cdot$ on the set $\mathbb{Z}$ of integers by $x + y = \max(x, y)$ and $x \cdot y = \min(x, y)$.
   a. Show that the commutative, associative, and distributive properties of a Boolean algebra hold for these two operations on $\mathbb{Z}$.
   b. Show that no matter what element of $\mathbb{Z}$ is chosen to be 0, the property $x + 0 = x$ of a Boolean algebra fails to hold.

4. Let $M_2(\mathbb{Z})$ denote the set of 2×2 matrices with integer entries, and let + denote matrix addition and $\cdot$ denote matrix multiplication. Given

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \mathbf{A}' = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$$

Using $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the 0 element and the 1 element, respectively, either prove that $[M_2(\mathbb{Z}), +, \cdot, ', 0, 1]$ is a Boolean algebra or give a reason why it is not.

5. Let S be the set $\{0, 1\}$. Then S^2 is the set of all ordered pairs of 0s and 1s; $S^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Consider the set B of all functions mapping S^2 to S. For example, one such function, $f(x,y)$, is given by

$$f(0, 0) = 0$$
$$f(0, 1) = 1$$
$$f(1, 0) = 1$$
$$f(1, 1) = 1$$

a. How many elements are in B?

b. For f_1 and f_2 members of B and $(x, y) \in S^2$, define

$$(f_1 + f_2)(x, y) = \max(f_1(x, y), f_2(x, y))$$
$$(f_1 \cdot f_2)(x, y) = \min(f_1(x, y), f_2(x, y))$$
$$f'(x, y) = \begin{cases} 1 \text{ if } f_1(x, y) = 0 \\ 0 \text{ if } f_1(x, y) = 1 \end{cases}$$

Suppose

$$\begin{array}{ll} f_1(0, 0) = 1 & f_2(0, 0) = 1 \\ f_1(0, 1) = 0 & f_2(0, 1) = 1 \\ f_1(1, 0) = 1 & f_2(1, 0) = 0 \\ f_1(1, 1) = 0 & f_2(1, 1) = 0 \end{array}$$

What are the functions $f_1 + f_2$, $f_1 \cdot f_2$, and f_1'?

c. Prove that $[B, +, \cdot, ', 0, 1]$ is a Boolean algebra where the functions 0 and 1 are defined by

$$\begin{array}{ll} 0(0, 0) = 0 & 1(0, 0) = 1 \\ 0(0, 1) = 0 & 1(0, 1) = 1 \\ 0(1, 0) = 0 & 1(1, 0) = 1 \\ 0(1, 1) = 0 & 1(1, 1) = 1 \end{array}$$

6. Let n be a positive integer whose decomposition into prime factors has no repeated prime. Let $B = \{x | x \text{ is a divisor of } n\}$. For example, if $n = 21 = 3 \cdot 7$, then $B = \{1, 3, 7, 21\}$. Let the following operations be defined on B:

$$x + y = \text{lcm}(x, y) \qquad x \cdot y = \text{gcd}(x, y) \qquad x' = n/x$$

Then $+$ and $\cdot$ are binary operations on B and $'$ is a unary operation on B.

a. For $n = 21$, find
   (i) $3 \cdot 7$
   (ii) $7 \cdot 21$
   (iii) $1 + 3$
   (iv) $3 + 21$
   (v) $3'$

b. Prove that the commutative, associative, and distributive properties hold for both $+$ and $\cdot$.

c. Find the value of the "0" element and the "1" element, then prove properties 4 and 5 for both $+$ and $\cdot$.

d. Consider a value for n whose decomposition has repeated primes. In particular, let $n = 12 = 2 \cdot 2 \cdot 3$. Prove that, using the above definitions for $+$ and $\cdot$, it's not possible to define a complement for 6 in the set $\{1, 2, 3, 4, 6, 12\}$. Therefore a Boolean algebra cannot be constructed with $n = 12$ using the process described.

7. Prove the following property of Boolean algebras. Give a reason for each step. (*Hint:* Remember the uniqueness of the complement.)

$$(x')' = x \qquad \text{(double negation)}$$

8. Prove the following property of Boolean algebras. Give a reason for each step. (*Hint:* Remember the uniqueness of the complement.)

$$(x + y)' = x' \cdot y' \qquad (x \cdot y)' = x' + y' \quad \text{(De Morgan's laws)}$$

9. Prove the following properties of Boolean algebras. Give a reason for each step.

a. $x + (x \cdot y) = x$ (absorption properties)
   $x \cdot (x + y) = x$

b. $x \cdot [y + (x \cdot z)] = (x \cdot y) + (x \cdot z)$ (modular properties)
   $x + [y \cdot (x + z)] = (x + y) \cdot (x + z)$

c. $(x + y) \cdot (x' + y) = y$
   $(x \cdot y) + (x' \cdot y) = y$

d. $(x + (y \cdot z))' = x' \cdot y' + x' \cdot z'$
   $(x \cdot (y + z))' = (x' + y') \cdot (x' + z')$

e. $(x + y) \cdot (x + 1) = x + (x \cdot y) + y$
   $(x \cdot y) + (x \cdot 0) = x \cdot (x + y) \cdot y$

10. Prove the following properties of Boolean algebras. Give a reason for each step.

a. $(x + y) + (y \cdot x') = x + y$

b. $(y + x) \cdot (z + y) + x \cdot z \cdot (z + z') = y + x \cdot z$

c. $(y' \cdot x) + x + (y + x) \cdot y' = x + (y' \cdot x)$

d. $(x + y') \cdot z = [(x' + z') \cdot (y + z')]'$

e. $(x \cdot y) + (x' \cdot z) + (x' \cdot y \cdot z') = y + (x' \cdot z)$

11. Prove the following properties of Boolean algebras. Give a reason for each step.
    a. $x + y' = x + (x' \cdot y + x \cdot y)'$
    b. $[(x \cdot y) \cdot z] + (y \cdot z) = y \cdot z$
    c. $x \cdot y + y \cdot x' = x \cdot y + y$
    d. $(x + y)' \cdot z + x' \cdot z \cdot y = x' \cdot z$
    e. $(x \cdot y') + (y \cdot z') + (x' \cdot z) = (x' \cdot y) + (y' \cdot z) + (x \cdot z')$
12. Prove the following properties of Boolean algebras. Give a reason for each step.
    a. $(x + y \cdot x)' = x'$
    b. $x \cdot (z + y) + (x' + y)' = x$
    c. $(x \cdot y)' + x' \cdot z + y' \cdot z = x' + y'$
    d. $x \cdot y + x' = y + x' \cdot y'$
    e. $x \cdot y + y \cdot z \cdot x' = y \cdot z + y \cdot x \cdot z'$
13. Prove that in any Boolean algebra, $x \cdot y' + x' \cdot y = y$ if and only if $x = 0$.
14. Prove that in any Boolean algebra, $x \cdot y' = 0$ if and only if $x \cdot y = x$.
15. A new binary operation $\oplus$ in a Boolean algebra (*exclusive OR*) is defined by

$$x \oplus y = x \cdot y' + y \cdot x'$$

    Prove that
    a. $x \oplus y = y \oplus x$
    b. $x \oplus x = 0$
    c. $0 \oplus x = x$
    d. $1 \oplus x = x'$
16. Prove that for any Boolean algebra:
    a. If $x + y = 0$, then $x = 0$ and $y = 0$.
    b. $x = y$ if and only if $x \cdot y' + y \cdot x' = 0$.
17. Prove that the 0 element in any Boolean algebra is unique; prove that the 1 element in any Boolean algebra is unique.
18. a. Find an example of a Boolean algebra with elements x, y, and z for which $x + y = x + z$ but $y \neq z$. (Here is further evidence that ordinary arithmetic of integers is not a Boolean algebra.)
    b. Prove that in any Boolean algebra, if $x + y = x + z$ and $x' + y = x' + z$, then $y = z$.
19. Let $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$ be two partially ordered sets. $(S, \preccurlyeq)$ is isomorphic to $(S', \preccurlyeq')$ if there is a bijection $f: S \to S'$ such that for x, y in S, $x \prec y \to f(x) \prec' f(y)$ and $f(x) \prec' f(y) \to x \prec y$.
    a. Show that there are exactly two nonisomorphic, partially ordered sets with two elements (use diagrams).
    b. Show that there are exactly five nonisomorphic, partially ordered sets with three elements.
    c. How many nonisomorphic, partially ordered sets with four elements are there?
20. Find an example of two partially ordered sets $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$ and a bijection $f: S \to S'$ where, for x, y in S, $x \prec y \to f(x) \prec' f(y)$ but $f(x) \prec' f(y) \nrightarrow x \prec y$.