EXAMPLE 36 The hunting grounds of a medieval king were forbidden to commoners, and anyone caught poaching the royal deer was subject to death. The hapless poacher was, however, granted a means to choose the manner of death. He (or she) was allowed to make a final statement. If the statement were judged to be true, death would be by beheading with a sword; if false, death would come by arrow shot from the bow of the best royal marksman. One day a particularly clever poacher was apprehended and allowed the usual final statement. The poacher said, "I will be shot to death by an arrow."

The king's court faced a conundrum. If the poacher were shot to death by an arrow, then the statement he made would prove to be true, in which case he should have been beheaded. But if he were beheaded, then the statement he made would be false, in which case he should have been shot by an arrow. Unable to decide the manner of death, the court appointed the clever poacher to the post of king's press secretary, where he served happily for many years.

This sort of *paradox*—a riddle with no solution—has to be carefully constructed, and we will not spend any more time reflecting on the potential shortcomings of classical logic systems that it may reveal. •

In addition to logical thinking in its pure sense, the notions of formal rules of inference have two very direct applications to computer science. An entire system of programming, and some programming languages, are based on applying rules of inference. We will see such a language in Section 1.5. Similarly, rules of inference can be applied to formally prove program correctness, leading to increased confidence that code is error-free. We'll look at some of the inference rules for program correctness in Section 1.6.

SECTION 1.4 REVIEW

TECHNIQUES

- Apply derivation rules for predicate logic.
- Use predicate logic to prove the validity of a verbal argument.

MAIN IDEA

- The predicate logic system is correct and complete; valid arguments and only valid arguments are provable.

EXERCISES 1.4

For Exercises 1–6, decide what conclusion, if any, can be reached from the given hypotheses and justify your answer.

1. All flowers are plants. Pansies are flowers.
2. All flowers are plants. Pansies are plants.
3. All flowers are red or purple. Pansies are flowers. Pansies are not purple.
4. Some flowers are purple. All purple flowers are small.
5 Some flowers are red. Some flowers are purple. Pansies are flowers.
6. Some flowers are pink and have thorns. All thorny flowers smell bad. Every flower that smells bad is a weed.

7. Justify each step in the following proof sequence of

$$(\exists x)[P(x) \to Q(x)] \to [(\forall x)P(x) \to (\exists x)Q(x)]$$

1. $(\exists x)[P(x) \to Q(x)]$
2. $P(a) \to Q(a)$
3. $(\forall x)P(x)$
4. $P(a)$
5. $Q(a)$
6. $(\exists x)Q(x)$

8. Justify each step in the following proof sequence of

$$(\exists x)P(x) \wedge (\forall x)(P(x) \to Q(x)) \to (\exists x)Q(x)$$

1. $(\exists x)P(x)$
2. $(\forall x)(P(x) \to Q(x))$
3. $P(a)$
4. $P(a) \to Q(a)$
5. $Q(a)$
6. $(\exists x)Q(x)$

9. Consider the wff

$$(\forall x)[(\exists y)P(x, y) \wedge (\exists y)Q(x, y)] \to (\forall x)(\exists y)[P(x, y) \wedge Q(x, y)]$$

a. Find an interpretation to prove that this wff is not valid.

b. Find the flaw in the following "proof" of this wff.

1.	$(\forall x)[(\exists y)P(x, y) \wedge (\exists y)Q(x, y)]$	hyp
2.	$(\forall x)[P(x, a) \wedge Q(x, a)]$	1, ei
3.	$(\forall x)(\exists y)[P(x, y) \wedge Q(x, y)]$	2, eg

10. Consider the wff

$$(\forall y)(\exists x)Q(x, y) \to (\exists x)(\forall y)Q(x, y)$$

a. Find an interpretation to prove that this wff is not valid.

b. Find the flaw in the following "proof" of this wff.

1.	$(\forall y)(\exists x)Q(x, y)$	hyp
2.	$(\exists x)Q(x, y)$	1, ui
3.	$Q(a, y)$	2, ei
4.	$(\forall y)Q(a, y)$	3, ug
5.	$(\exists x)(\forall y)Q(x, y)$	4, eg

In Exercises 11–16, prove that each wff is a valid argument.

11. $(\forall x)P(x) \to (\forall x)[P(x) \vee Q(x)]$
12. $(\forall x)P(x) \wedge (\exists x)Q(x) \to (\exists x)[P(x) \wedge Q(x)]$
13. $(\exists x)(\exists y)P(x, y) \to (\exists y)(\exists x)P(x, y)$
14. $(\forall x)(\forall y)Q(x, y) \to (\forall y)(\forall x)Q(x, y)$

15. $(\forall x)P(x) \wedge (\exists x)[P(x)]' \rightarrow (\exists x)Q(x)$
16. $(\forall x)[S(x) \rightarrow (\exists y)(P(x,y) \wedge T(y))] \wedge (\exists x)(C(x) \wedge S(x)) \rightarrow (\exists x)(\exists y)(C(x) \wedge T(y) \wedge P(x, y))$

In Exercises 17–30, either prove that the wff is a valid argument or give an interpretation in which it is false.

17. $(\exists x)[A(x) \wedge B(x)] \rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$
18. $(\exists x)[R(x) \vee S(x)] \rightarrow (\exists x)R(x) \vee (\exists x)S(x)$
19. $(\exists x)P(x) \wedge (\exists x)(\exists y)Q(x, y) \rightarrow (\exists x)(\exists y)[P(x) \wedge Q(x, y)]$
20. $(\forall x)[P(x) \rightarrow Q(x)] \rightarrow [(\forall x)P(x) \rightarrow (\forall x)Q(x)]$
21. $(\forall x)(P(x))' \rightarrow (\forall x)(P(x) \rightarrow Q(x))$
22. $[(\forall x)P(x) \rightarrow (\forall x)Q(x)] \rightarrow (\forall x)[P(x) \rightarrow Q(x)]$
23. $(\exists x)(\forall y)Q(x, y) \rightarrow (\forall y)(\exists x)Q(x, y)$
24. $(\forall x)P(x) \vee (\exists x)Q(x) \rightarrow (\forall x)[P(x) \vee Q(x)]$
25. $(\forall x)[A(x) \rightarrow B(x)] \rightarrow [(\exists x)A(x) \rightarrow (\exists x)B(x)]$
26. $(\forall y)[Q(x, y) \rightarrow P(x)] \rightarrow [(\exists y)Q(x, y) \rightarrow P(x)]$
27. $[P(x) \rightarrow (\exists y)Q(x, y)] \rightarrow (\exists y)[P(x) \rightarrow Q(x, y)]$
28. $(\forall x)(P(x) \vee Q(x)) \wedge (\exists x)Q(x) \rightarrow (\exists x)P(x)$
29. $(\exists x)[P(x) \wedge Q(x)] \wedge (\forall y)[Q(y) \rightarrow R(y)] \rightarrow (\exists x)[P(x) \wedge R(x)]$
30. $(\forall x)(\forall y)[(P(x) \wedge S(x, y)) \rightarrow Q(y)] \wedge (\exists x)B(x) \wedge (\forall x)(B(x) \rightarrow P(x)) \wedge (\forall x)(\exists y)S(x, y) \rightarrow (\exists x)Q(x)$
31. The Greek philosopher Aristotle (384–322 B.C.E.) studied under Plato and tutored Alexander the Great. His studies of logic influenced philosophers for hundreds of years. His four "perfect" syllogisms are identified by the names given them by medieval scholars. For each, formulate the argument in predicate logic notation and then provide a proof.
    a. "Barbara"
       All *M* are *P*
       All *S* are *M*
       Therefore all *S* are *P*
    b. "Celarent"
       No *M* are *P*
       All *S* are *M*
       Therefore no *S* are *P*
    c. "Darii"
       All *M* are *P*
       Some *S* are *M*
       Therefore some *S* are *P*
    d. "Ferio"
       No *M* are *P*
       Some *S* are *M*
       Therefore some *S* are not *P*

Using predicate logic, prove that each argument in Exercises 32–42 is valid. Use the predicate symbols shown.

32. Some plants are flowers. All flowers smell sweet. Therefore, some plants smell sweet. $P(x)$, $F(x)$, $S(x)$

33. Every crocodile is bigger than every alligator. Sam is a crocodile. But there is a snake, and Sam isn't bigger than the snake. Therefore, something is not an alligator. $C(x)$, $A(x)$, $B(x, y)$, s, $S(x)$
34. There is an astronomer who is not nearsighted. Everyone who wears glasses is nearsighted. Furthermore, everyone either wears glasses or wears contact lenses. Therefore, some astronomer wears contact lenses. $A(x)$, $N(x)$, $G(x)$, $C(x)$
35. Every member of the board comes from industry or government. Everyone from government who has a law degree is in favor of the motion. John is not from industry, but he does have a law degree. Therefore, if John is a member of the board, he is in favor of the motion. $M(x)$, $I(x)$, $G(x)$, $L(x)$, $F(x)$, j
36. There is some movie star who is richer than everyone. Anyone who is richer than anyone else pays more taxes than anyone else does. Therefore, there is a movie star who pays more taxes than anyone. $M(x)$, $R(x, y)$, $T(x, y)$
37. Everyone with red hair has freckles. Someone has red hair and big feet. Everybody who doesn't have green eyes doesn't have big feet. Therefore someone has green eyes and freckles. $R(x)$, $F(x)$, $B(x)$, $G(x)$
38. Cats eat only animals. Something fuzzy exists. Everything that's fuzzy is a cat. And everything eats something. So animals exist. $C(x)$, $E(x, y)$, $A(x)$, $F(x)$
39. Every computer science student works harder than somebody, and everyone who works harder than any other person gets less sleep than that person. Maria is a computer science student. Therefore, Maria gets less sleep than someone else. $C(x)$, $W(x, y)$, $S(x, y)$, m
40. Every ambassador speaks only to diplomats, and some ambassador speaks to someone. Therefore, there is a diplomat. $A(x)$, $S(x, y)$, $D(x)$
41. Some elephants are afraid of all mice. Some mice are small. Therefore there is an elephant that is afraid of something small. $E(x)$, $M(x)$, $A(x, y)$, $S(x)$
42. Every farmer owns a cow. No dentist owns a cow. Therefore no dentist is a farmer. $F(x)$, $C(x)$, $O(x, y)$, $D(x)$
43. Prove that

$$[(\forall x)A(x)]' \leftrightarrow (\exists x)[A(x)]'$$

is valid. (*Hint:* Instead of a proof sequence, use Example 32 and substitute equivalent expressions.)

44. The equivalence of Exercise 43 says that if it is false that every element of the domain has property A, then some element of the domain fails to have property A, and vice versa. The element that fails to have property A is called a counterexample to the assertion that every element has property A. Thus a counterexample to the assertion

$$(\forall x)(x \text{ is odd})$$

in the domain of integers is the number 10, an even integer. (Of course, there are lots of other counterexamples to this assertion.) Find counterexamples in the domain of integers to the following assertions. (An integer $x > 1$ is prime if the only factors of x are 1 and x.)

a. $(\forall x)(x \text{ is negative})$
b. $(\forall x)(x \text{ is the sum of even integers})$
c. $(\forall x)(x \text{ is prime} \rightarrow x \text{ is odd})$
d. $(\forall x)(x \text{ prime} \rightarrow (-1)^x = -1)$
e. $(\forall x)(x \text{ prime} \rightarrow 2^x - 1 \text{ is prime})$