Section 3.1: Topology of the Real Numbers

November 15, 2011

Abstract

The material we discuss in this section is quite general, but applications to the real numbers are the focus. You are already familiar with open and closed intervals – this section includes more details about the conditions under which a set will be open or closed, including some rather interesting additional concepts such as boundary points, interior points, limit points, etc.

Our author states that "...our primary purpose for studying topology is to enable us to study properties of continuous functions."

Definition 3-1: open: A set U of real numbers is said to be *open* if, for each $x \in U$, $\exists \delta(x) > 0$ such that $(x - \delta(x), x + \delta(x)) \subset U$.

Observation: This means that every element of A has a buffer-zone around it: that is, is surrounded entirely by elements of A. No point is "exposed" to non-A points.

Theorem 3-1: : The intervals (a, b), (a, ∞), and $(-\infty, a)$ are open sets.

Definition 3-2: closed: A set A is said to be *closed* if A^c is open.

Corollary 3-1: The intervals $(-\infty, a]$, $[a, b]$, and $[a, \infty)$ are closed sets.

Theorem 3-2: The open sets of real numbers satisfy the following conditions:

(i) if $\{U_1, \ldots, U_n\}$ is a **finite collection** of open sets, then $\cap_{k=1}^{n} U_k$ is an open set.

(ii) if $\{U_\alpha\}$ is **any collection** of open sets, then $\cup_\alpha U_\alpha$ is an open set.

Note: So we can use these first two theorems as a recursive definition of open sets: the base cases are in Theorem 3-1, and the inductive step is in Theorem 3-2. We can quickly generate new open sets (e.g. $\mathbb{R}$, and $\emptyset$), using the base cases and unions and intersections.

Note: The important thing to notice is that only **finite** intersections are open, whereas arbitrary unions are open. While it is possible to have an infinite intersection of open sets that is open, it is not necessarily the case: we can have an infinite intersection of open sets that even ends up closed: consider, for example, the intervals

$$U_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) \qquad \forall n \in \mathbb{N}$$

$$\bigcap_{n=1}^{\infty} U_n = [0, 1]$$

Theorem 3-3: The closed sets satisfy the following properties:

(i) $\emptyset$ and $\mathbb{R}$ are closed.

(ii) if $\{A_\alpha\}$ is **any collection** of closed sets, then $\cap_\alpha A_\alpha$ is closed.

(iii) if $\{A_1, \ldots, A_n\}$ is a **finite collection** of closed sets, then $\cup_{k=1}^n A_k$ is closed.

Note: By contrast, only **finite** unions are close, whereas arbitrary intersections are closed. While it is possible to have an infinite union of closed sets that is closed, it is not necessarily the case: we can have an infinite union of closed sets that even ends up open: consider, for example, the closed sets of single points

$$U_x = \{x\} \qquad \forall x \in (0, 1)$$

Then

$$\bigcup_x U_x = (0, 1)$$

Observation: There are also sets that are neither open nor closed, of course: e.g. $[0, 1)$. But the building blocks are open and closed sets.

Definition 3-3: open relative to A: Let A be a set of real numbers. The set V is *open relative to A* if

$$V = A \cap U$$

for some open set U. Notice that $V \subset A$. This definition is motivated by the need to restrict the universe from the reals to a smaller domain in the case of many functions. Then we want to speak of sets open relative to this diminished universe.

Theorem 3-4: Let A be a set of real numbers. Then the subsets of A that are open relative to A satisfy the following conditions:

(i) $\emptyset$ and A are open.

(ii) if $\{V_1, \ldots, V_n\}$ is a finite collection of sets that are open relative to A, then $\cap_{k=1}^n V_k$ is open relative to A.

(iii) if $\{V_\alpha\}$ is any collection of sets that are open relative to A, then $\cup_\alpha V_\alpha$ is open relative A.

Note: Notice that these are the same results as expressed in Theorem 3-2.

Theorem 3-5: A set is open iff it can be expressed as a countable union of disjoint open invervals.

Definition 3-4: interior point, boundary point, limit point, isolated point: Let A be a set of real numbers.

(i) If there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$, then x is said to be an *interior point* of A. The set consisting of all the interior points of A is called the *interior of* A and is denoted $int(A)$.

Note: $x \in A$

(ii) If $\forall \delta > 0$, the interval $(x - \delta, x + \delta)$ contains a point in A and a point not in A, then x is said to be a *boundary point* of A. The set of all boundary points of A is called the *boundary of* A and is denoted $b(A)$.

Note: x is not necessarily in A. Furthermore, x may be the only point in a δ-neighborhood of x (i.e. isolated – see below).

(iii) If $\forall \delta > 0$, the interval $(x - \delta, x + \delta)$ contains a point of A distinct from x, then x is said to be a *limit point* (or *cluster point*, or *accumulation point*) of A.

Note: x is not necessarily in A.

(iv) A point $x \in A$ is said to be an *isolated point of* A if there is a $\delta > 0$ such that $(x - \delta, x + \delta) \cap A = \{x\}$.

Note: $x \in A$, and x is a boundary point (albeit out on the frontier, shall we say).

Theorem 3-6: A set is closed iff it contains all of its boundary points.

Note: Every boundary point is a limit point.

Corollary 3-6:

- A set is closed iff it contains all of its limit points.
- A set is open iff it contains none of its boundary points (that is, all limit points are interior points).

Definition 3-5: closure: Let A be a set of real numbers. The *closure of* A, denoted $\overline{A}$, is the set consisting of A and its limit points.

Theorem 3-7: Let A be a set of real numbers. Then $\overline{A}$ is a closed set.

Note: The closure is well named!

Definition 3-6: cover, open cover, finite cover: Let A be a set. The collection of sets $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be a *cover* of A if $A \subset \cup_{\alpha\in\mathcal{A}} I_\alpha$. If each set I_α is open, the collection is said to be an *open cover* of A. If the number of sets in the collection $\mathcal{A}$ is finite, then the collection is said to be a *finite cover.*

Definition 3-7: compact: A set A is said to be *compact* if every open cover of A has a finite subcover.

Note: If the set is compact, we can always get away with **less**: every open cover has a finite subcover (that is, we can throw out all but finitely many members of the cover).

Theorem 3-8: Let $\{A_1, A_2, \ldots\}$ be a countable collection of nonempty, closed bounded sets of real numbers such that $A_i \supset A_j$ if $i \leq j$. Then $\cap A_i \neq \emptyset$.

Corollary 3-8: Let $\{A_1, A_2, \ldots\}$ be a countable collection of closed bounded sets of real numbers such that $A_i \supset A_j$ if $i < j$. If $\cap_{i=1}^{\infty} A_i = \emptyset$, then there is a positive integer N such that $\cap_{i=1}^{N} A_i = \emptyset$.

Theorem 3-9: Let A be a set of real numbers. Then if $\{I_\alpha\}$ is an open cover of A, some *countable* subcollection of $\{I_\alpha\}$ covers A.

Theorem 3-10 (Heine-Borel Theorem): If A is a closed bounded set of real numbers, then A is compact.

Theorem 3-11: A set that is compact is closed and bounded.

Theorem 3-12: A set A of real numbers is compact iff every infinite set of points in A has a limit point in A.

Theorem 3-13: Let A be a set of real numbers. The following are equivalent.

(i) A is compact.

(ii) A is closed and bounded.

(iii) Every infinite set of points in A has a limit point in A.

(iv) If $\{x_n\}$ is a sequence of numbers in A, there is a subsequence $\{x_{n_k}\}$ that converges to some point in A.

Definition 3-8: connected: A set A is said to be *connected* if there do not exist two open sets U and V such that

(i) $U \cap V = \emptyset$.

(ii) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$.

(iii) $(U \cap A) \cup (V \cap A) = A$.

Note: It is interesting that this definition is given in terms of a negative – a property that it does not possess.

Theorem 3-14: A set of real numbers with more than one element is connected iff it is an interval.

Example: Exercise 10ab, p. 71

(i) Show that $int(A)$ is open.

By definition, $int(A) = \{x \in A | \exists \delta(x) > 0 \ / \ (x - \delta(x), x + \delta(x)) \subset A\}$.

By contradiction. Assume that $int(A)$ is not open. Then $\exists \alpha \in A$ such that there does not exist $\delta(\alpha) > 0 \ / \ (\alpha - \delta(\alpha), \alpha + \delta(\alpha)) \subset B$.

But $\exists \delta(\alpha) > 0 \ / \ (\alpha - \delta(\alpha), \alpha + \delta(\alpha)) \subset A$. Hence $\forall x \in (\alpha - \frac{\delta(\alpha)}{2}, \alpha + \frac{\delta(\alpha)}{2})$, $x \in B$ (take $\delta(x) = \frac{\delta(\alpha)}{2}$ - then $(x - \delta(x), x + \delta(x)) \subset (\alpha - \delta(\alpha), \alpha + \delta(\alpha)) \subset A$, so $x \in B$. Thus $(\alpha - \delta(x), \alpha + \delta(x)) \subset B$, a contradiction (we assumed that no such open interval existed for α).

(ii) B open $\wedge$ $B \subset A \rightarrow B \subset int(A)$

Directly: assume B open $\wedge$ $B \subset A$. Then $\forall x \in B$, $\exists \delta(x) > 0 / (x - \delta(x), x + \delta(x)) \subset B \subset A$, hence $x \in int(A)$.

Therefore $x \in B \wedge B \subset A \rightarrow x \in int(A)$. QED.

Observation: One observation about compactness and open covers. One student argued in a problem set that because the set $A = [0, 1]$ is compact, every open cover has a finite subcover. Then the student argued that if $B \subset A$, then every open cover of B has a finite subcover – one can simply refer to the finite subcover of A! Alas, this isn't right – if it were, then $B = (0, 1)$ would be compact (and it's not, as an open set). The trick is that B has an open cover that A doesn't have. For example,

$$B = \bigcup_{i=1}^{\infty} \left(\frac{1}{n+1}, 1 \right)$$

And it's clear that there is no open subcover. It's interesting to think about the impact of covering the endpoints: it provides a cushion around 0, $(0 - \varepsilon, 0 + \varepsilon)$, that allows us to get away with only a finite number of the sets in the union above.