Section 4.2: Monotone and Inverse Functions

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Abstract

One of the interesting topics we need to consider more is the nature of **dis**-continuity. In addition, standard topics like one-sided limits and continuity will be considered, as well as the possible discontinuities of monotone functions. Finally, we'll see how these are applied in the case of invertibility.

1 Limits Involving Infinity

Definition 4-6: infinite limits: Let f be a function with domain $D(f)$, and suppose that x_0 is a limit point of $D(f)$. We say that *the limit of f as x approaches x_0 is ∞* if, given any $M \in \mathbb{R}$ there is a number $\delta(M) > 0$ such that if $0 < |x - x_0| < \delta(M)$ and $x \in D(f)$, then $f(x) > M$.

Definition 4-7: finite limit as $x \to \infty$: Let f be a function with domain $D(f)$ such that $D(f) \cap (M, \infty) \neq \emptyset$ for any number M. Then we say that *the limit of f as x goes to ∞ is L* provided that, if $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ such that $x > N(\varepsilon)$ and $x \in D(f)$, then $|f(x) - L| < \varepsilon$.

Definition 4-8: infinite limit as x approaches ∞: Let f be a function with domain $D(f)$ such that $D(f) \cap (M, \infty) \neq \emptyset$ for any number M. We say that *the limit of f as x apaproaches ∞ is ∞* provided that, $\forall K$, $\exists N(K)$ such that $x > N(K)$ and $x \in D(f)$, then $f(x) > K$.

2 Right- and Left-Hand Limits

Definition 4-9: one-sided limits from the right: Let f be a function with domain $D(f)$, and suppose that x_0 is a limit point of $D(f) \cap [x_0, \infty)$. We say that *the limit of f as x approaches x_0 from the right is L* if, $\forall \varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $\forall x \in D(f)$

$$0 < x - x_0 < \delta(\varepsilon) \to |f(x) - L| < \varepsilon.$$

In this case, we write $\lim_{x \downarrow x_0} = L$ or $\lim_{x \to x_0^+} = L$.

Definition 4-10: one-sided limits from the left: Let f be a function with domain $D(f)$, and suppose that x_0 is a limit point of $(-\infty, x_0] \cap D(f)$. We say that *the limit of f as x approaches x_0 from the left is L* if, $\forall \varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $\forall x \in D(f)$

$$0 < x_0 - x < \delta(\varepsilon) \rightarrow |f(x) - L| < \varepsilon.$$

In this case, we write $\lim_{x \uparrow x_0} = L$ or $\lim_{x \to x_0^-} = L$.

Theorem 4-11: Let f be a function with $D(f)$, and suppose x_0 is a limit point of $D(f) \cap [x_0, \infty)$ and $D(f) \cap (-\infty, x_0]$. Then $\lim_{x \to x_0} f(x) = L$ if and only if

$$\lim_{x \uparrow x_0} = L \qquad \text{and} \qquad \lim_{x \downarrow x_0} = L.$$

Definition 4-11: f is continuous from the right (left) at x_0: Let f be a function with domain $D(f)$, and suppose $x_0 \in D(f)$. We say that f *is continuous from the right (left) at x_0* if, given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that if $0 \leq x - x_0 < \delta(\varepsilon)$ $(0 \leq x_0 - x < \delta(\varepsilon))$ and $x \in D(f)$, then $|f(x) - f(x_0)| < \varepsilon$.

Corollary 4-11: Let f be a function with domain $D(f)$, and suppose $x_0 \in D(f)$. Then f is continuous at x_0 if and only if it is continuous from the right and from the left at x_0.

Theorem 4-12: Let f be a function with domain $D(f)$, and suppose $x_0 \in D(f)$. Then f is continuous from the right (left) at x_0 if and only if for every sequence $\{x_n\} \subset D(f)$ with $x_n \geq x_0$ $(x_0 \leq x_n)$ and $\{x_n\} \to x_0$, $\{f(x_n)\} \to f(x_0)$.

Corollary 4-12: Let f be a function with domain $D(f)$, and suppose $x_0 \in D(f)$. Then f is continuous from the right (left) at x_0 if and only if for every decreasing (increasing) sequence $\{x_n\} \subset D(f)$ with $\{x_n\} \to x_0$, $\{f(x_n)\} \to f(x_0)$.

3 Monotone Functions

Definition 4-12: Monotone increasing (decreasing): A function is said to be **monotone increasing (decreasing)** if for any $x_1, x_2 \in D(f)$ with $x_1 < x_2$, $f(x_1) \leq f(x_2)$ $(f(x_1) \geq f(x_2))$. If, for any $x_1, x_2 \in D(f)$ with $x_1 < x_2$, $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$, f is **strictly monotone increasing (decreasing)**. A function that is either monotone increasing or decreasing is said to be monotone.

Theorem 4-13: Let f be a monotone function with domain an open interval (a, b). Then $\lim_{x \uparrow x_0} f(x)$ and $\lim_{x \downarrow x_0} f(x)$ exist for each $x_0 \in (a, b)$.

Theorem 4-14: A monotone function f with $D(f) \supset (a, b)$ can have at most countably many discontinuities on (a, b).

4 Types of Discontinuities

Definition 4-13: removable discontinuity: A function f is said to have a **removable discontinuity** at x_0 if $\lim_{x \to x_0} f(x)$ exists, but $\lim_{x \to x_0} f(x) \neq f(x_0)$ or $f(x_0)$ does not exist.

Definition 4-14: jump discontinuity: A function f is said to have a **jump discontinuity** at x_0 if $\lim_{x \uparrow x_0} f(x)$ exists, and $\lim_{x \downarrow x_0} f(x)$ exists, but they don't agree.

Definition 4-15: discontinuity of the third type: A function f is said to have a **discontinuity of the third type** at x_0 if f fails to be continuous at x_0, but it is neither a removable nor jump discontinuity.

5 Continuity of the Inverse Function

Main result: the inverse of a continuous 1-1 function on a closed interval is continuous.

Theorem 4-16: Suppose f is continuous and strictly monotonic on $[a, b]$. Then

(i) f^{-1} is strictly monotone on its domain;

(ii) f^{-1} is continuous on its domain.