
On Forming Committees

P. Haxell

Abstract. This paper surveys a collection of results on finding special sets of vertices in graphs with vertex partitions, all of which can be viewed as models for “committee-choosing” problems.

1. INTRODUCTION. As usual, the dean of your institution wishes to form a committee, consisting of one representative from each department in the faculty. Choosing such a committee without any restrictions would be an easy task; however, as everybody knows, there are certain pairs of faculty members who really should not be on a committee together. Neither one of them can tolerate the other having the last word. Thus in order to ensure finite committee meetings, the dean should avoid choosing both members of such a pair when selecting the members of the committee. In this article we are interested in finding out when this is possible.

First we describe the problem in terms of graphs. Let G be a graph, and suppose the *vertex set* of G (the set of faculty members) is partitioned into *classes* (departments). Each *edge* of G is a pair of vertices representing a conflict, i.e., a pair of faculty members who should not both be chosen for the committee. Note that since we will choose only one representative from each department, we may assume that no two vertices in the same class are joined by an edge. An *independent transversal* (abbreviated IT) of G is a set T of vertices in G containing exactly one vertex from each class of the partition (a *transversal*) that is *independent*, meaning no two vertices in T are joined by an edge. In other words, an independent transversal in G is exactly the notion of a good choice of committee from the faculty.

It only takes a moment to see the first piece of bad news for the dean: that faculties can exist that simply do not have any good choice of committee. Imagine for example a faculty in which no member of the Department of Environmental Studies, however open-minded, could see eye to eye on any issue with any member of the Department of Mountaintop-Removal Mining Development. (This corresponds to a *complete bipartite graph* between two partition classes in G .) Thus we might ask whether there is at least an easy way to decide whether a given partitioned graph has an IT or not.

This brings us to the second piece of bad news. Consider the restricted version of the committee-choosing problem in which each faculty member is captivated by one very important two-sided issue. Two people holding opposite views on the same issue simply can't be on a committee together. Let us suppose further that no department contains two members who care about the same issue. This situation then corresponds to the graph G being a disjoint union of complete bipartite graphs, in which each complete bipartite graph contains at most one vertex of each partition class. Can we efficiently determine if a good committee exists? Unfortunately almost certainly not, as this is a model of the most basic NP-complete problem, the SAT problem (see, e.g., [15]). Suppose A is a boolean formula in conjunctive normal form, that is, A is a conjunction of clauses, each of which is a disjunction of propositional variables and negations of variables. Let G be the graph in which each clause of A corresponds to a partition class of G , and the vertices of the partition class are labelled with the variables and negations of variables that occur in the clause (see Figure 1). For each variable x ,

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join each vertex labelled x to each vertex labelled with the negation of x . (Thus each “issue” is represented by a variable.) Then in the resulting partitioned graph G , an independent transversal corresponds exactly to a satisfying truth assignment for A .

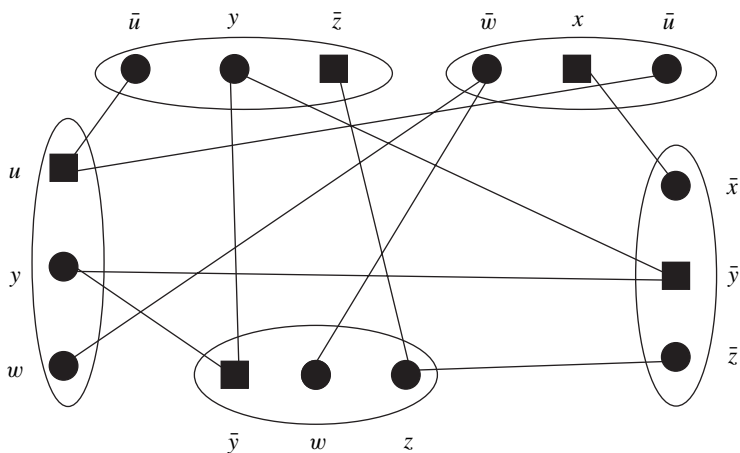


Figure 1. The graph corresponding to the boolean formula $A = (u \vee y \vee w) \wedge (\bar{u} \vee y \vee \bar{z}) \wedge (\bar{w} \vee x \vee \bar{u}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (\bar{y} \vee w \vee z)$. The square vertices form an IT corresponding to the truth assignment $u = x = T$, $y = z = F$, and w can be either T or F .

What help can we then offer to the dean? The best we can do is to provide some sufficient conditions for an IT to exist in a given vertex-partitioned graph that are not too hard to check. In the next section we show that if the partition classes are big enough compared to the maximum degree of the graph, then an IT always exists. In Section 4 we describe a somewhat more complicated condition for the existence of an IT, which nevertheless can be applied in certain circumstances. Here we will appeal to Sperner’s lemma, a result from combinatorial topology.

A very optimistic dean might even want to form many disjoint committees, perhaps even imagining a perfect world in which the entire faculty could be partitioned into disjoint committees. This “happy dean” problem is captured by the notion of strong colouring, which we discuss in Section 3. We end the paper with some remarks and pointers to other related topics.

2. MAXIMUM DEGREE (LIMITED PERSONAL CONFLICT). In this section we discover that if no faculty member conflicts with too many others, then the dean can choose a good committee as long as each department is large enough. Recall that for a vertex x of a graph G , the *degree* $d(x)$ of x in G is the number of edges incident to x . We denote by $\Delta(G)$ the maximum degree of G ; in other words, no faculty member conflicts with more than Δ others.

Theorem 2.1. *Let G be a graph with a vertex partition. Suppose each partition class has size at least $2\Delta(G)$. Then G has an IT.*

Theorem 2.1 first appeared explicitly in [17], and has been applied in a number of other settings, for example [2, 8, 11, 25]. It is best possible, as Szabó and Tardos [29] gave constructions of graphs G with partition classes of size $2\Delta(G) - 1$ that do not have independent transversals (see also Bollobás, Erdős, and Szemerédi [10], Jin [23], and Yuster [30] for earlier constructions for certain values of Δ). A more precise

version of Theorem 2.1, giving a bound in terms of Δ and the number of partition classes m , was given in [20] (see also [7]).

We will derive Theorem 2.1 from a more general result, which gives information about the structure of *edge-minimal* graphs that do not have independent transversals. For a graph G with vertex set $V(G)$ and edge set $E(G)$, and a set of vertices $W \subseteq V(G)$, the *neighbourhood* of W is $\Gamma(W) = \{y : wy \in E(G) \text{ for some } w \in W\}$. When $W = \{w\}$ we use the abbreviation $\Gamma(w)$ for $\Gamma(\{w\})$. We say W *dominates* every vertex in $\Gamma(W)$, and we say W *dominates* G if $\Gamma(W) = V(G)$. We use the notation $G[W]$ to denote the subgraph of G induced by W , that is, the graph with vertex set W and edge set $\{xy \in E(G) : x, y \in W\}$. For a set Z of edges we write $v(Z)$ for the set of vertices incident to edges of Z . Finally for a vertex x in a vertex-partitioned graph we denote by $V(x)$ the vertex class containing x . Our proof below follows that of [9].

Theorem 2.2. *Let G be a graph, and suppose $V_1 \cup \dots \cup V_m$ is a partition of $V(G)$ into m independent vertex classes. Suppose G has no IT, but for every edge e the graph $G - e$, formed by removing e from G , has an IT. Let $e = xy \in E(G)$. Then there exists a subset $S \subseteq \{V_1, \dots, V_m\}$ and a set of edges Z of $G_S = G[\bigcup_{V_i \in S} V_i]$ such that*

1. $V(x), V(y) \in S$ and $e \in Z$,
2. $v(Z)$ dominates G_S ,
3. $|Z| \leq |S| - 1$.

To see that Theorem 2.2 implies Theorem 2.1, let G be as in Theorem 2.1 and suppose on the contrary that it has no IT. Remove edges one by one from G until the resulting graph satisfies the assumptions of Theorem 2.2, and let S be the subset of classes given by Theorem 2.2. The number of vertices that can be dominated by $v(Z)$, a set of size at most $2|S| - 2$, is at most $(2|S| - 2)\Delta$, but G_S contains $2\Delta|S|$ vertices. Thus conclusions (2) and (3) cannot hold, so this contradiction shows that G must have had an IT.

Proof. We prove Theorem 2.2 by induction on m . Let G and $e = xy$ be as in the statement of the theorem. The assertion of the theorem is trivially true when $m = 1$, so assume $m \geq 2$ and that the statement is true for smaller values of m .

Choose an IT T of $G - e$. Then $x, y \in T$, since otherwise T would be an IT of G . We form a new graph H by

- removing the vertex set $W = \Gamma(\{x, y\})$ from G (note $x, y \in W$), and
- unifying the remaining vertices in $V(x) \cup V(y)$ into one new vertex class Y^* (and removing any edges inside Y^*).

Each class V_i other than $V(x)$ and $V(y)$ just becomes $Y_i = V_i \setminus W$ in H . Note that each class apart from possibly Y^* is nonempty because it still contains an element of T .

Case 1: $Y^* = \emptyset$.

In this case set $S = \{V(x), V(y)\}$ and $Z = \{e\}$. Then $v(Z) = \{x, y\}$ dominates all of G_S as required.

Case 2: $Y^* \neq \emptyset$.

First we verify that H does not have an IT. Suppose on the contrary that T' is an IT for H . Let z be the vertex of T' in Y^* . Then by definition of Y^* , in G we have either $z \in V(x)$ or $z \in V(y)$. But then in the first case, by definition of H the set $\{y\} \cup T'$ is an IT of G , and in the second case $\{x\} \cup T'$ is an IT of G . This contradiction shows that H has no IT.

Now we remove edges one by one from H until we obtain a graph H' with no IT but such that the removal of any edge results in a graph with an IT. Note that each vertex w in Y^* has degree at least one in H' , since otherwise $T \setminus \{x, y\} \cup \{w\}$ would be an IT of H' . Let e' be any edge of H' incident to a vertex of Y^* . Then since H' has $m - 1$ vertex classes, by induction there exists a set S' of vertex classes (containing Y^*) together with a set of edges Z' in H'_S that satisfies the conclusions (1)–(3). Note that since H' was obtained from H by removing edges we also know that $v(Z')$ dominates H'_S . But then setting $S = S' \setminus \{Y^*\} \cup \{V(x), V(y)\}$ and $Z = Z' \cup \{e\}$ gives the required conclusion. ■

In fact it follows from the above proof that the set Z of edges is a *matching*, that is, no two edges in Z share a vertex.

3. STRONG COLOURING (THE HAPPY DEAN PROBLEM). Suppose the dean does indeed have a good choice of committee. What could be better than finding many committees, say even a partition of the entire faculty into disjoint committees (assuming, unrealistically here, that the dean wouldn't expect any faculty member to be on more than one committee at once)? This delightful situation for the dean is related to the notion of strong colouring in graphs. A partition of the vertex set $V(G)$ into independent sets (committees) I_1, \dots, I_r is called a *proper colouring* of the graph G with the r colours $1, \dots, r$, a very well-studied notion in graph theory. Here however the dean also requires something extra, namely that each class (department) contains exactly one member of each colour (committee); see Figure 2.

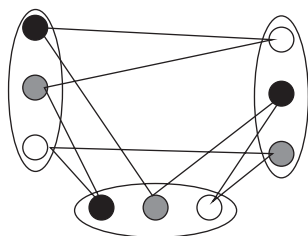


Figure 2. A partition into 3 committees (the white, grey, and black committees) for a faculty with 3 departments.

Let r and n be positive integers such that r divides n , and let G be a graph with n vertices. We call G *strongly r -colourable* if for every partition of the vertex set $V(G)$ into parts V_k of size r , there exists a proper colouring of G with r colours with the additional property that each V_k contains exactly one vertex of each colour. If r does not divide n then we say G is strongly r -colourable if the graph obtained by adding $r \lceil n/r \rceil - n$ isolated vertices to G is strongly r -colourable. It can be shown that if G is strongly r -colourable then it is also strongly $(r + 1)$ -colourable. The *strong chromatic number* $s\chi(G)$ of a graph G is defined to be the minimum r such that G is strongly r -colourable. This notion was introduced independently by Alon [6] and Fellows [13].

Thus the dean could be guaranteed a partition of the entire faculty G into committees provided the departments have size at least $s\chi(G)$ (and, of course, they all have the same size). If the unlucky dean finds that the faculty is exactly the graph given in [29] (see Section 2) with departments of size $2\Delta(G) - 1$ but no independent transversal, then not even one committee can be found, let alone a partition into committees. Thus for some graphs $s\chi(G) \geq 2\Delta(G)$. It is conjectured that $2\Delta(G)$ is in fact also an upper bound for every G , but the best known bounds for strong colouring

in terms of $\Delta(G)$ are $s\chi(G) \leq 3\Delta(G) - 1$ for every graph [18], and asymptotically $s\chi(G) \leq (1 + o(1))11\Delta(G)/4$ [19].

Here we give a short proof (by Aharoni, Berger, and Ziv [2], based on [18]) that $s\chi(G) \leq 3\Delta(G)$ for every G . We begin with a slight extension of Theorem 2.1 that says that in fact, if all departments have size at least $2\Delta(G)$, then the dean can even choose one specific person on the faculty in advance (who is especially suitable to chair this committee, say), and still find a good committee that contains this person.

Theorem 3.1. *Let G be a vertex-partitioned graph of maximum degree Δ in which each partition class has size at least 2Δ . Let $x \in V(G)$. Then G has an IT containing x .*

Proof. Let V_1, \dots, V_m denote the vertex classes of G (again we may remove any edge that is inside a class). Assume without loss of generality that $x \in V_1$. Let G' be the graph, with vertex classes $V'_i \subseteq V_i$ for $2 \leq i \leq m$, obtained by removing V_1 and all neighbours of x from G . Note that each V'_i is nonempty since $|V_i| \geq 2\Delta$. If G' has an IT T then $T \cup \{x\}$ is an IT of G as required. If G' has no IT then remove edges from G' until every remaining edge prevents an IT. Let e be an arbitrary edge of the resulting graph H (which must exist since each class of H is nonempty). We apply Theorem 2.2 to H and e to obtain a subset S of classes and a set of edges Z of size at most $|S| - 1$ such that $v(Z)$ dominates H_S . But $v(Z)$ can dominate at most $2\Delta|Z| \leq 2\Delta(|S| - 1)$ vertices, and H_S contains at least $2\Delta|S| - \Delta$ vertices. This contradiction shows that H has an IT, and hence G has an IT containing x . ■

We represent a proper colouring of G with colours $1, \dots, r$ by a function $\alpha : V(G) \rightarrow \{1, \dots, r\}$, where $\{x : \alpha(x) = j\}$ is independent for each j .

Theorem 3.2. *Let G be a graph of maximum degree Δ . Then $s\chi(G) \leq 3\Delta$.*

Proof. By adding isolated vertices as necessary, we may assume $n = |V(G)|$ is divisible by 3Δ . Let a partition $\mathcal{P} = V_1 \cup \dots \cup V_m$ of $V(G)$ into classes of size 3Δ be fixed, and suppose on the contrary that there is no suitable colouring of G with respect to \mathcal{P} . Fix a *maximum partial colouring* (MPC) α of G , that is, a proper colouring using 3Δ colours of as many vertices of G as possible such that no two vertices in the same partition class have the same colour. Suppose V_1 is a class that contains an uncoloured vertex x . Then some colour in $\{1, \dots, 3\Delta\}$ is not used in V_1 ; let us call this colour *red*. For each i , $2 \leq i \leq m$, let r_i denote the red vertex in V_i (if it exists). Our plan is to “swap colours” between each r_i and another vertex in the same class as r_i . For $2 \leq i \leq m$ set $W_i = V_i \setminus \{v : \alpha(v) = \alpha(z) \text{ for some } z \in \Gamma(r_i)\}$ (if r_i does not exist then $W_i = V_i$), so W_i is the set of vertices whose colour could be given to r_i . Set also $W_1 = V_1$. Then each $|W_i| \geq 2\Delta$, so by Theorem 3.1 the graph $G[W_1 \cup \dots \cup W_m]$ has an IT T with $x \in T$. Modify α by giving colour red to every vertex of T , and for each i for which r_i exists and $r_i \notin T$, give r_i the colour of the element t_i of T in V_i . Then since T was an IT, and by definition of the W_i , this gives a valid colouring α' . Moreover α' colours all the vertices that were coloured by α together with x , contradicting the fact that α was an MPC. Therefore a valid colouring must exist for \mathcal{P} , and thus we conclude $s\chi(G) \leq 3\Delta$. ■

Finding the correct function of $\Delta(G)$ that bounds $s\chi(G)$ from above seems to be difficult (see Problem 4.14 in Jensen and Toft [22]). For example, it is not even known whether $s\chi(G) \leq 4$ for every graph of maximum degree two. By [18] we know $s\chi(G) \leq 5$ for such graphs.

4. A HALL-TYPE CONDITION. The criterion for the existence of an IT that we describe in this section uses a rather more complicated notion than maximum degree or class size, so we begin with the relevant definition. Let G be a graph, and suppose $V_1 \cup \dots \cup V_m$ is a partition of $V(G)$ into m independent vertex classes. Let $S \subseteq \{V_1, \dots, V_m\}$ be a subset of classes. We call an independent set I_S of vertices of $G_S = G[\bigcup_{V_i \in S} V_i]$ *special* for S if for every independent subset J of vertices of G_S with $|J| \leq |S| - 1$, there exists $v \in I_S$ such that $J \cup \{v\}$ is also independent. We can think of a special independent set as a “neutral team” for the set S of departments: a group of mutually nonconflicting members from those departments that can augment any “small” set of mutually nonconflicting members. (Here “augment” is not quite the right word, since if I_S actually contains an element v of J then $J \cup \{v\}$ is of course the same size as J and satisfies the condition.) Note that the departments from which the members of I_S or J are taken do not figure in this definition.

The reason we refer to the upcoming theorem as a Hall-type condition for the existence of an IT is by analogy with Hall’s classical theorem for the existence of a matching of size $|A|$ in a bipartite graph H with vertex classes A and X (here *bipartite* means $V(H) = A \cup X$, and A and X are disjoint independent sets in H).

Theorem 4.1. (*Hall’s Theorem*) *A bipartite graph H with vertex classes A and X has a matching of size $|A|$ if and only if for every subset S of A we have $|\Gamma(S)| \geq |S|$.*

We may now state the main theorem of this section, which is from [4]. Note that in contrast to Hall’s theorem this is only a one-way implication; for an if-and-only-if version see Section 5 (where we also show that this theorem implies Hall’s theorem).

Theorem 4.2. *Let G be a graph with a vertex partition into independent classes. Suppose that for every subset S of classes, the graph G_S contains an independent set I_S that is special for S . Then G has an IT.*

The dean may be justifiably dubious that this theorem could possibly be useful, as checking whether a partitioned graph satisfies the condition (of having a special independent set for every subset of classes) looks hopelessly complicated. Thus to convince the dean to read on, we first give a quick application of Theorem 4.2 to a well-known combinatorial problem. Then in the following subsection we will describe the proof, which uses Sperner’s lemma. We mention that another application of Theorem 4.2 (and in fact its original motivation) is an extension of Hall’s theorem to hypergraphs (see [4]).

4.1. Application of Theorem 4.2. Our application is related to the famous “cycle plus triangles” problem, popularised by Erdős in the 1980s, and finally solved by Fleischner and Stiebitz [14] in 1992 and with a different proof by Sachs [28] in 1993. It asks whether every union of a cycle C_{3k} of length divisible by three, together with a set of k disjoint triangles on the same vertex set, has a proper colouring with three colours (see Figure 3). In the language of Section 3, this is the same as asking whether every C_{3k} is strongly 3-colourable. This question of Erdős was motivated by an earlier question of Du, Hsu, and Hwang [12], who asked whether every C_{3k} with an arbitrary vertex partition into classes of size three has an IT. This problem was also unsolved until the proofs of Fleischner and Stiebitz, and Sachs, of the stronger statement. Both of these proofs are ingenious but quite difficult. Here we can give a solution to the question of Du, Hsu, and Hwang that is almost immediate from Theorem 4.2, and in fact is more general.

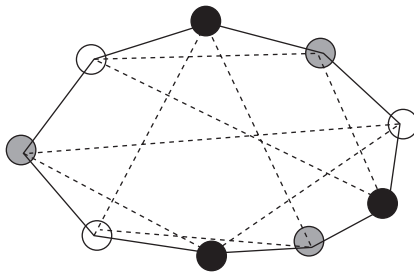


Figure 3. A 3-coloured cycle-plus-triangles graph on the cycle C_9 . Compare to Figure 2, which is a different drawing of C_9 , with the same three triangles represented as departments.

Theorem 4.3. *Let G be a graph of maximum degree two, in which each cycle is of length divisible by three. Then G has an IT with respect to any vertex partition into classes of size at least three.*

Proof. Fix a vertex partition, and remove any edge that is inside a class. Let S be any subset of classes. Then the graph G_S has at least $3|S|$ vertices, and its components are cycles of length divisible by three and paths. Let I_S be a maximum independent set in G_S that is spaced three apart. For example, I_S could be formed by taking, from each path component, an end vertex and then every third vertex starting from that end, and, from each cycle component C_{3k} , an independent set of size k containing every third vertex. We claim that I_S is special for S . To see this, let J be an arbitrary independent set in G_S of size $|S| - 1$. Since I_S is spaced three apart, each vertex of J is adjacent to at most one vertex of I_S . Thus since $|I_S| \geq |S|$, there is a vertex $v \in I_S$ such that $J \cup \{v\}$ is independent. Thus I_S is special, and so by Theorem 4.2 we know that G has an IT. ■

4.2. Proof of Theorem 4.2. First we recall Sperner’s lemma, a basic result from combinatorial topology. Suppose F is a triangulation of the $(m - 1)$ -dimensional simplex Σ_{m-1} . For each point x of F we denote by $f(x)$ the face of Σ_{m-1} containing x in its interior. A labelling of the points of F with elements of $\{1, \dots, m\}$ is called a *Sperner labelling* if

- each vertex of Σ_{m-1} receives a different label, and
- each point x of F receives the same label as some vertex of $f(x)$.

An example of a Sperner labelling is shown in Figure 4. We call a simplex of the triangulation *fully-labelled* if it receives all m labels on its vertices.

Theorem 4.4. (*Sperner’s Lemma*) *Let F be a triangulation of Σ_{m-1} with a Sperner labelling. Then the number of fully-labelled simplices in F is odd.*

As is usually the case, the implication of Sperner’s lemma that is useful for us is that the number of fully-labelled simplices is nonzero. Note that there are three fully-labelled simplices in Figure 4.

For the proof of Theorem 4.2 we also need to know that certain special triangulations of Σ_{m-1} exist (see Figure 4). The *1-skeleton* of a triangulation F is the graph whose vertices are the points of F , and whose edges are the 1-dimensional simplices of F .

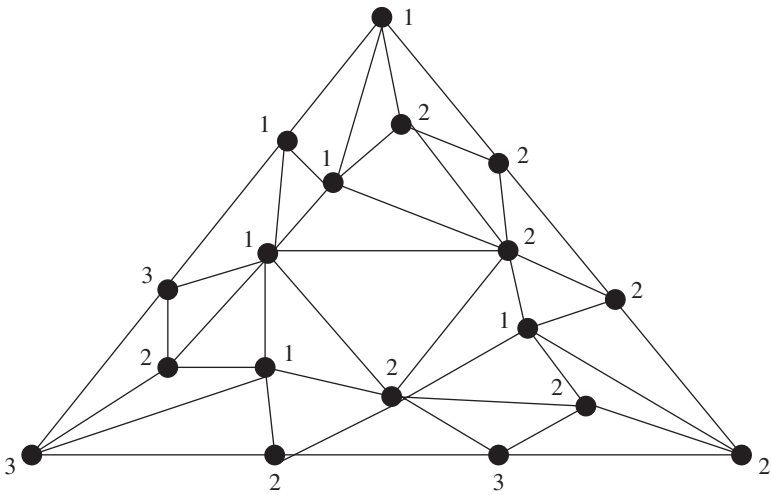


Figure 4. A Sperner labelling of a triangulation F of Σ_2 with the properties in Lemma 4.5.

Lemma 4.5. *There exists a triangulation F of Σ_{m-1} with the following properties.*

- (i) *If x and y are adjacent in the 1-skeleton of F then one of $f(x)$ and $f(y)$ contains the other,*
- (ii) *if x has neighbours in the 1-skeleton of F on the boundary of $f(x)$, then these neighbours are the vertices of a simplex of F .*

A proof of Lemma 4.5 can be found in [4]. A different construction was given in [3].

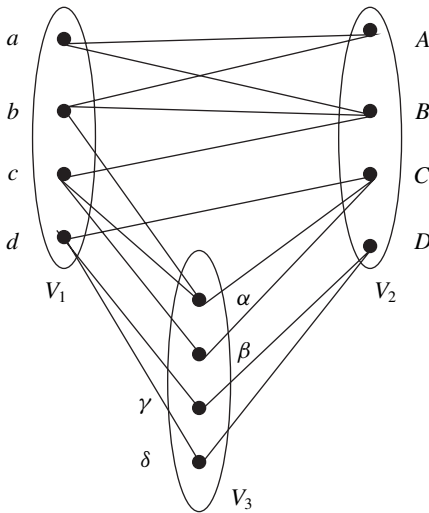


Figure 5. The graph G .

Proof. We may now give the proof of Theorem 4.2. Let the graph G with vertex partition $V_1 \cup \dots \cup V_m$ be given (see Figure 5 for an example). Let F be the triangulation of Σ_{m-1} given by Lemma 4.5, and let z_1, \dots, z_m denote the vertices of Σ_{m-1} . We define a function g that assigns to each point of F a vertex of G with the following properties.

- (1) For each point x , the vertex $g(x)$ is an element of I_S where $S = \{V_i : z_i \text{ is a vertex of } f(x)\}$.
- (2) If points x and y are adjacent in the 1-skeleton of F then $g(x)$ and $g(y)$ are not joined by an edge in G .
- (3) The labelling ℓ of F defined by $\ell(x) = i$ where $g(x) \in V_i$ is a Sperner labelling.

A suitable function g for the graph G in Figure 5 is shown in Figure 6.

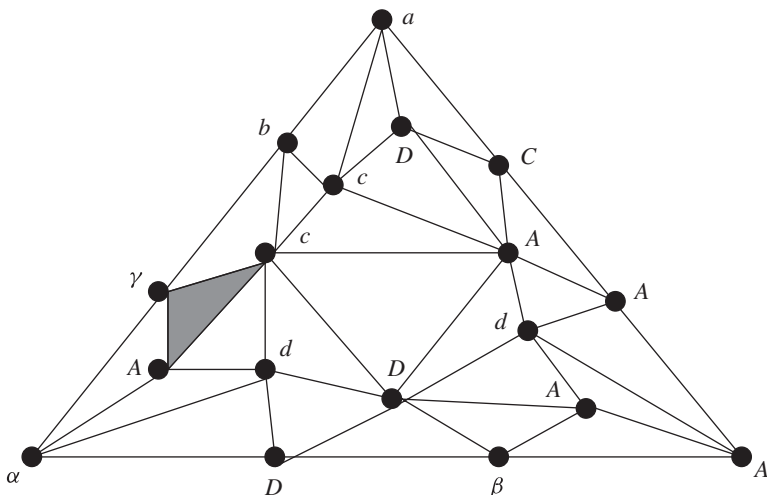


Figure 6. The triangulation F with the vertex $g(z)$ of G assigned to each point z .

Our motivation here is as follows. Once we have found g , by (3), the labelling ℓ is a Sperner labelling. (The labelling ℓ for the function g in Figure 6 is the labelling in Figure 4.) Therefore by Theorem 4.4 there exists a simplex W of F that gets all m labels on its vertices (e.g., the shaded triangle in Figure 6). By (2), the set of vertices $I = \{g(x) : x \in W\}$ is independent in G . By definition of ℓ , each $v \in I$ lies in a distinct class V_i . Therefore I is an IT of G as required. ($I = \{c, A, \gamma\}$ in our example.)

Thus to finish the proof, we define g on the points x of F , in increasing order of $\dim(f(x))$. For the points in zero-dimensional faces (i.e., the vertices z_1, \dots, z_m of Σ_{m-1}) we choose an arbitrary vertex $v_i \in I_{\{V_i\}}$ (which is nonempty by the assumption of the theorem) and set $g(z_i) = v_i$. This satisfies (1), satisfies (2) by Lemma 4.5(i), and is consistent with (3).

Now suppose $j \geq 1$ and we have defined g on all points in faces of dimension smaller than j , and possibly some in faces of dimension j , such that (1)–(3) are satisfied (see Figure 7). Let x be in a face f of dimension j (e.g., the grey point in Figure 7), and let $S = \{V_i : z_i \text{ is a vertex of } f\}$. ($S = \{V_1, V_2, V_3\}$ in Figure 7.) By Lemma 4.5(i) and (ii), if x has any neighbours in the 1-skeleton of F in faces of dimension smaller than j , then the set U of such neighbours lies in the boundary of f and forms a simplex of F . Thus $|U| \leq \dim(f) = |S| - 1$ and by (2) $J = \{g(x) : x \in U\}$ is independent in G . ($J = \{b, \gamma\}$ in Figure 7.) Thus J is a “small” independent set, and so the special independent set I_S contains a vertex v such that $J \cup \{v\}$ is independent. (In our example suppose $I_S = \{c, d, A, D\}$. Then $v = c$ is a suitable choice.) Then we set $g(x) = v$, so (1) and (3) are satisfied. To check (2), if y is a neighbour of x in a smaller dimensional face then (2) is satisfied by choice of v . If y is in a face of dimension j (e.g., the white point in Figure 7) then by Lemma 4.5(i) it must be in f as well. Thus if $g(y)$ has

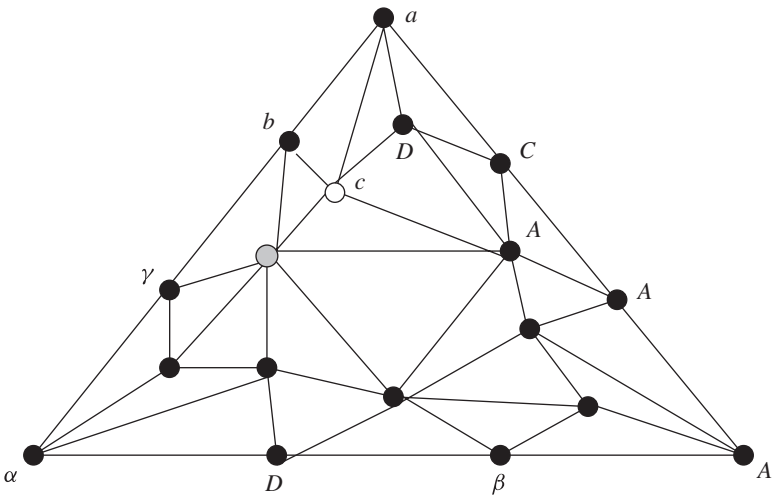


Figure 7. Defining g on the points of F .

already been defined then $g(y) \in I_S$ by (1) ($g(y) = c$ in our example) and therefore is not adjacent to $g(x)$ in G . This completes the definition of g , and hence the proof. ■

5. REMARKS. The if-and-only-if version of Theorem 4.2 is as follows.

Theorem 5.1. *Let G be a graph, and suppose $V_1 \cup \dots \cup V_m$ is a partition of $V(G)$ into m independent vertex classes. Then G has an IT if and only if the following holds: for each $S \subseteq \{V_1, \dots, V_m\}$ there exists an independent set I_S of vertices of $G_S = G[\bigcup_{V_i \in S} V_i]$ such that for every independent subset $J \subseteq \bigcup_{U \subset S} I_U$ with $|J| \leq |S| - 1$, there exists $v \in I_S$ such that $J \cup \{v\}$ is also independent.*

The “only-if” implication is easy: just take $I_S = I \cap V(G_S)$ for an IT I . The proof of the “if” implication is the same as the proof of Theorem 4.2. To see that Hall’s theorem is a special case, suppose a bipartite graph H as in Theorem 4.1 is given, such that $|\Gamma(S)| \geq |S|$ for every $S \subseteq A$. Define a graph G whose vertex set is the set of edges of H by joining e and f by an edge of G if and only if e and f are incident to the same vertex of X in H . The vertices of G are then partitioned into $|A|$ classes, according to the vertex of A they are incident to in H . Thus an IT of G is precisely a matching of size $|A|$ in H . To verify the assumption of Theorem 5.1, let S be a subset of the vertex classes. Choose I_S to be a set of $|S|$ edges of H , all incident to distinct vertices in X . This is possible by Hall’s condition on H . Then for any set J of edges of H of size at most $|S| - 1$, some element e of I_S has a different X -vertex from all elements of J , and thus $J \cup \{e\}$ is also independent. Therefore G has an IT.

The use of topological arguments for transversal-type problems has been taken much further; see, e.g., [26], [29], and [1]. In [1], Aharoni and Berger obtain wide-ranging results in the much more general setting of matroids, which have many interesting applications.

The dean now knows that good committees exist in many situations, but suddenly wants to know something else: how do we actually find these committees? Are there efficient algorithms for finding them? Each of the proofs of the theorems we have seen (Theorems 2.2, 3.2, and 4.2) does in fact give a procedure for finding the IT that the theorem claims exists. Unfortunately though, in each case, the number of steps

is potentially exponential in the size of the graph. Thus, sadly, we can't reassure the dean that finding these committees is always an easy task. In general, the problem of finding structures that are guaranteed to exist but are apparently hard to find gives rise to a fascinating branch of complexity theory; see for example [27].

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PENNY HAXELL received her B.Math from the University of Waterloo in 1988 and her Ph.D. from the University of Cambridge in 1993. She has been a faculty member in the Combinatorics and Optimization department at the University of Waterloo since 1993. This article is based on an MAA invited address at the AMS-MAA Joint Meetings in New Orleans in 2007. The author has no good excuse for taking this long to write it.

Department of Combinatorics and Optimization, University of Waterloo, Waterloo Ont. Canada N2L 3G1.
pehaxell@math.uwaterloo.ca

Mathematics Is ... and Is Not ...

“Mathematics is abstract thought, mathematics is pure logic, mathematics is creative art.”

Paul Halmos, *Mathematics as a creative art*,
American Scientist **56**(4) (1968) 380.

“Mathematics is not a deductive science—that’s a cliché. When you try to prove a theorem, you don’t just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.”

Paul Halmos, *I Want to Be a Mathematician: An Automathography*,
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